Chapter 1

# IDENTIFYING $Q$-PROCESSES WITH A GIVEN FINITE $\mu$-INVARIANT MEASURE

P.K. Pollett
*Department of Mathematics*
*The University of Queensland*
*Queensland 4072*
*Australia*
pkp@maths.uq.edu.au

**Abstract** Let $Q = (q_{ij},\ i,j \in S)$ be a stable and conservative $Q$-matrix over a state space $S$ consisting of an irreducible (transient) class $C$ and a single absorbing state 0, which is accessible from $C$. Suppose that $Q$ admits a *finite* $\mu$-subinvariant measure $m = (m_j, j \in C)$ on $C$. We consider the problem of identifying all $Q$-processes for which $m$ is a $\mu$-invariant measure on $C$.

**Keywords:** $Q$-processes; quasistationary distributions; construction theory

## 1. INTRODUCTION

[1]We begin with a totally stable $Q$-matrix over a countable set $S$, that is, a collection $Q = (q_{ij},\ i,j \in S)$ of real numbers which satisfies

$$\begin{aligned} 0 \le q_{ij} < \infty, &\quad i \ne j,\ i,j \in S, \\ q_i := -q_{ii} < \infty, &\quad i \in S, \\ \sum_{j \ne i} q_{ij} \le q_i, &\quad i \in S. \end{aligned} \tag{1.1}$$

The $Q$-matrix is said to be *conservative* if equality holds in (1.1) for all $i \in S$. For simplicity, we shall assume that $Q$ is conservative. A set of real-valued functions $P(\cdot) = (p_{ij}(\cdot),\ i,j \in S)$ defined on $(0,\infty)$ is called a *standard*

[1]This manuscript was prepared using the Edited Book LaTeX style file ©Kluwer Academic Press 1998.

*transition function* (or *process*) if

$$p_{ij}(t) \geq 0, \qquad i,j \in S,\ t > 0, \tag{1.2}$$

$$\sum_{j \in S} p_{ij}(t) \leq 1, \qquad i \in S,\ t > 0, \tag{1.3}$$

$$p_{ij}(s+t) = \sum_{k \in S} p_{ik}(s) p_{kj}(t), \quad i,j \in S,\ s,t > 0, \tag{1.4}$$

$$\lim_{t \downarrow 0} p_{ij}(t) = \delta_{ij}, \qquad i,j \in S. \tag{1.5}$$

$P$ is then *honest* if equality holds in (1.3) for some (and then all) $t > 0$, and it is called a *$Q$-transition function* (or *$Q$-process*) if $p'_{ij}(0+) = q_{ij}$ for each $i,j \in S$. Under the conditions we have imposed, every $Q$-process $P$ satisfies the *backward differential equations,*

$$p'_{ij}(t) = \sum_{k \in S} q_{ik} p_{kj}(t), \qquad t > 0, \tag{BE$_{ij}$}$$

for all $i,j \in S$, but might not satisfy the *forward differential equations*,

$$p'_{ij}(t) = \sum_{k \in S} p_{ik}(t) q_{kj}, \qquad t > 0, \tag{FE$_{ij}$}$$

for all $i,j \in S$. The classical construction problem is to find one and then all $Q$-processes. Feller's recursion (Feller, 1940) provides for the existence of a *minimal* solution $F(\cdot) = (f_{ij}(\cdot),\ i,j \in S)$ to the backward equations (which also satisfies the forward equations); see also (Feller, 1957; Reuter, 1957). This process is the *unique* $Q$-process if and only if the system of equations

$$\sum_{j \in S} q_{ij} x_j = \nu x_i, \qquad i \in S, \tag{1.6}$$

has no bounded, non-trivial solution (equivalently, non-negative solution) $x$ for some (and then all) $\nu > 0$ (Reuter, 1957); for the non-conservative case, see (Hou, 1974). When this condition fails, there are infinitely many $Q$-processes, including infinitely many honest ones (Reuter, 1957), and the dimension $d$ of the space of bounded vectors $x$ on $S$ satisfying (1.6), a quantity which does not depend on $\nu$, determines the number of "escape routes to infinity" available to the process. A construction of all $Q$-processes was given by (Reuter, 1959; Reuter, 1962) under the assumption that $d = 1$ (the *single-exit case*), and this was later extended to the *finite-exit case* ($d < \infty$) by (Williams, 1964).

If (1.6) has infinitely many bounded non-trivial solutions, the problem of constructing all $Q$-processes remains unsolved; there are simply too many solutions of the backward equations to characterize. For this reason, variants of the classical construction have been considered in which various side conditions are imposed. The most recent work centres on an assumption that one is given an *invariant measure* for the $Q$-matrix, that is, a collection of positive numbers $m = (m_i,\ i \in S)$ which satisfy

$$\sum_{i \in S} m_i q_{ij} = 0, \qquad j \in S.$$

The problem is then to identify $Q$-processes with $m$ as their invariant measure, that is

$$\sum_{i \in S} m_i p_{ij}(t) = m_j, \qquad j \in S,\ t > 0.$$

When does there exist such a $Q$-process, and, when is it a unique $Q$-process with the given invariant measure? This variant of the classical construction problem has particular significance when $m$ is finite ($\sum m_i < \infty$), for then one is looking for a $Q$-process whose *stationary distribution* has been specified. The problem of existence, and then uniqueness in the single-exit case, was solved by (Hou and Chen, 1980) under the assumption that $Q$ is $m$-*symmetrizable*, that is,

$$m_i q_{ij} = m_j q_{ji}, \qquad i, j \in S,$$

(see (Chen and Zhang, 1987) for the non-conservative case) and by me in the general case (Pollett, 1991a; Pollett, 1994). Recently Han-jun Zhang announced a solution to the existence problem under more general circumstances; see (Zhang et al., 1998a; Zhang et al., 1998b).

In this paper we shall look at a slightly different kind of construction problem, where the state space can be decomposed into an irreducible class $C$ and a single absorbing state, and we shall suppose, rather than an *invariant* measure, a $\mu$-invariant measure on $C$ is specified through $Q$. We seek to determine $Q$-processes for which $m$ is a $\mu$-invariant measure on $C$. Since here we shall assume that the $\mu$-invariant measure is *finite*, we are effectively identifying $Q$-processes with a given *quasi-stationary distribution* (van Doorn, 1991). And, since we will not necessarily require these processes to satisfy the forward equations, we shall relax the $\mu$-invariance for $Q$ to $\mu$-*sub*invariance for $Q$.

Before proceeding, let me remark that in this introductory section I have restricted my attention to the *totally stable case* ($q_i < \infty$ for all $i \in S$). Of course, the problem of constructing $Q$-processes when all states, or a finite subset of states, are unstable is an important one, and can be traced back to Lévy and Kolmogorov; for an informative summary see (Rogers and Williams, 1986).

## 2. PRELIMINARIES

We shall suppose that $S = \{0\} \cup C$, where $C$ is an irreducible class (for the minimal $Q$-process, and hence for any $Q$-process) and 0 is an absorbing state which is accessible from $C$, that is $q_0 = 0$ and $q_{i0} > 0$ for at least one $i \in C$. Then, if $\mu$ is some fixed non-negative real number, a collection of strictly positive numbers $m = (m_j,\ j \in C)$ is called a *$\mu$-subinvariant measure (on $C$) for $Q$* if

$$\sum_{i \in C} m_i q_{ij} \le -\mu m_j, \qquad j \in C, \tag{1.7}$$

and *$\mu$-invariant* if equality holds for all $j \in C$. We shall suppose that $Q$ admits a $\mu$-subinvariant measure on $C$, and then identify $Q$-processes $P$ such that $m$ is a *$\mu$-invariant (on $C$) for $P$*, that is,

$$\sum_{i \in C} m_i p_{ij}(t) = e^{-\mu t} m_j, \qquad j \in C,\ t > 0. \tag{1.8}$$

The relationship between (1.7) and (1.8) has been divined completely for the minimal $Q$-process $F$. It was shown by (Tweedie, 1974) that if $m$ is a $\mu$-invariant measure for $F$, then it is $\mu$-invariant for $Q$. Conversely (Pollett, 1986; Pollett, 1988), if $m$ is a $\mu$-invariant measure for $Q$, then it is $\mu$-invariant for $F$ if and only if the equations

$$\sum_{i \in C} y_i q_{ij} = -\nu y_j, \qquad 0 \le y_j \le m_j,\ j \in C,$$

have no non-trivial solution for some (and then all) $\nu < \mu$. If $\mu > 0$ and the measure $m$ is assumed to be *finite*, that is $\sum_{i \in C} m_i < \infty$, then much simpler conditions obtain (Pollett and Vere-Jones, 1992; Nair and Pollett, 1993). For example, if $F$ is *honest* (and hence the unique $Q$-process), then a finite $\mu$-subinvariant measure $m$ for $Q$ is $\mu$-invariant for $F$ *if and only if*

$$\mu \sum_{i \in C} m_i = \sum_{i \in C} m_i q_{i0}. \tag{1.9}$$

As we shall see, this condition guarantees, more generally, that there *exists* a $Q$-process $P$ such that $m$ is a $\mu$-invariant measure for $P$; it is honest and satisfies (FE$_{i0}$) for $i \in C$. We note that, in determining such a $P$, we are effectively identifying a $Q$-process with a given *quasi-stationary distribution* (van Doorn, 1991): a probability distribution $\pi = (\pi_j,\ j \in C)$ over $C$ is called a quasi-stationary distribution if $p_j(t)/\sum_{i \in C} p_i(t) = \pi_j$ for all $t > 0$, where $p_j(t) = \sum_{i \in C} \pi_i p_{ij}(t)$, $t > 0$, so that, conditional on non-absorption, the state probabilities of the underlying continuous-time Markov chain are stationary. It was shown by (Nair and Pollett, 1993) that a distribution $\pi = (\pi_j,\ j \in C)$ is a quasi-stationary distribution if and only if, for some $\mu > 0$, $\pi$ is a $\mu$-invariant measure for $P$, in which case if $P$ is honest, then $p_{i0}(t) \to 1$ for all $i \in C$ as $t \to \infty$ (absorption occurs with probability 1).

## 3. THE MAIN RESULT

We shall specify transition functions through their resolvents. If $P$ is a given transition function, then the function $\Psi(\cdot) = (\psi_{ij}(\cdot),\ i,j \in S)$ given by

$$\psi_{ij}(\alpha) = \int_0^\infty e^{-\alpha t} p_{ij}(t)dt, \qquad \alpha > 0, \tag{1.10}$$

is called the *resolvent of* $P$. If $i,j \in C$, the integral in (1.10) converges for all $\alpha > -\lambda_P(C)$, where $\lambda_P(C)$ is the *decay parameter of* $C$ (for $P$); see (Kingman, 1963). In particular, since $C$ is irreducible, the integral (1.10) has the same abcissa of convergence for each $i,j \in C$. Notice also that, since 0 is an absorbing state, $\psi_{0j}(\alpha) = \delta_{0j}/\alpha$. Analogous to properties (1.2)–(1.5), $\Psi$ satisfies

$$\psi_{ij}(\alpha) \geq 0, \qquad i,j \in S,\ \alpha > 0, \tag{1.11}$$

$$\sum_{j\in S} \alpha\psi_{ij}(\alpha) \leq 1, \qquad i \in S,\ \alpha > 0, \tag{1.12}$$

$$\psi_{ij}(\alpha) - \psi_{ij}(\beta) + (\alpha - \beta)\sum_{k\in S} \psi_{ik}(\alpha)\psi_{kj}(\beta) = 0, \qquad i,j \in S,\ \alpha,\beta > 0, \tag{1.13}$$

$$\lim_{\alpha\to\infty} \alpha\psi_{ij}(\alpha) = \delta_{ij}, \quad i,j \in S. \tag{1.14}$$

(Note that (1.13) is called the *resolvent equation*.) Indeed, any $\Psi$ which satisfies (1.11)–(1.14) is the resolvent of a standard transition function (see (Reuter, 1959; Reuter, 1962)). Further, (1.12) is satisfied with equality if and only if $P$ is honest, in which case the *resolvent* is said to be honest. Also, the $Q$-matrix of $P$ can be recovered from $\Psi$ using the following identity:

$$q_{ij} = \lim_{\alpha\to\infty} \alpha(\alpha\psi_{ij}(\alpha) - \delta_{ij}). \tag{1.15}$$

And, a resolvent which satisfies (1.15) is called a $Q$-resolvent. The resolvent $\Phi(\cdot) = (\phi_{ij}(\cdot),\ i,j \in S)$ of the minimal $Q$-process has itself a minimal interpretation (see (Reuter, 1957; Reuter, 1959)); it is the minimal solution to the equations

$$\alpha\psi_{ij}(\alpha) = \delta_{ij} + \sum_{k\in S} q_{ik}\psi_{kj}(\alpha), \qquad i,j \in S,\ \alpha > 0,$$

which are analogous to (BE$_{ij}$), and $\Phi$ is called the minimal $Q$-resolvent.

We can identify $\mu$-invariant measures using resolvents. If $P$ is a $Q$-process with resolvent $\Psi$ and $m = (m_j,\ j \in C)$ is a $\mu$-invariant measure for $P$, where of necessity $\mu \leq \lambda_P(C)$ (see Lemma 4.1 of (Vere-Jones, 1967)), then, since the integral in (1.10) converges for all $\alpha > -\lambda_P(C)$, we have, for all $j \in C$ and $\alpha > 0$, that

$$\sum_{i \in S} m_i \alpha \psi_{ij}(\alpha - \mu) = m_j. \tag{1.16}$$

We refer to $m$ as being $\mu$-invariant for $\Psi$ if (1.16) is satisfied. Finally, a simple extension of Lemma 4.1 of (Pollett, 1991b) establishes that $m$ is $\mu$-invariant for $\Psi$ if it is $\mu$-invariant for $P$, and, if $\mu \le \lambda_P(C)$, then $m$ is $\mu$-invariant for $P$ if it is $\mu$-invariant for $\Psi$.

We are now ready to state our main result.

**Theorem 1** *Let $\mu > 0$ and suppose that $Q$ admits a finite $\mu$-subinvariant measure. Then, if*

$$\mu \sum_{i \in C} m_i = \sum_{i \in C} m_i q_{i0}, \tag{1.17}$$

*there exists a $Q$-process $P$ for which $m$ is $\mu$-invariant. The resolvent $\Psi(\cdot) = (\psi_{ij}(\cdot),\ i,j \in S)$ of one such $Q$-process has the form*

$$\psi_{ij}(\alpha) = \phi_{ij}(\alpha) + \frac{z_i(\alpha) d_j(\alpha)}{(\alpha + \mu) \sum_{k \in C} m_k z_k(\alpha)}, \qquad i,j \in S, \tag{1.18}$$

*where $z(\cdot) = (z_i(\cdot),\ i \in C)$ is given by*

$$z_i(\alpha) = 1 - \sum_{j \in S} \alpha \phi_{ij}(\alpha), \qquad i \in C,$$

*with the interpretation that $\Psi = \Phi$ if $z$ is identically $0$, and $d(\cdot) = (d_i(\cdot),\ i \in S)$ is given by*

$$d_i(\alpha) = m_i - \sum_{j \in C} m_j (\alpha + \mu) \phi_{ji}(\alpha), \qquad i \in C, \tag{1.19}$$

$$d_0(\alpha) = \frac{\mu}{\alpha} \sum_{j \in C} m_j - \sum_{j \in C} m_j (\alpha + \mu) \phi_{j0}(\alpha). \tag{1.20}$$

*This process is honest and satisfies* ($\mathrm{FE}_{i0}$) *for $i \in C$.*

*Proof*. First observe that if $z$ is identically 0, the minimal $Q$-process $F$ is honest and, by Theorem 3 of (Pollett and Vere-Jones, 1992), (1.17) is necessary and sufficient for $m$ to be $\mu$-invariant for $F$ (in which case $d$ is identically 0 and, by Proposition 2 of (Tweedie, 1974), $m$ is $\mu$-invariant for $Q$). Trivially, $F$ satisfies ($\mathrm{FE}_{i0}$) for $i \in C$.

Suppose that $z$ is not identically 0. We will first show that $m$ cannot be $\mu$-invariant for $F$ and, in so doing, establish that $d$ is not identically 0. Suppose, by contradiction, that $m$ *is* $\mu$-invariant for $F$, so that

$$\sum_{i \in C} m_i \phi_{ij}(\alpha) = \frac{m_j}{\alpha + \mu}, \qquad j \in C. \tag{1.21}$$

Multiplying by $\alpha$ and summing over $j \in C$ gives

$$\sum_{i\in C} m_i \alpha \phi_{i0}(\alpha) + \sum_{i\in C} m_i z_i(\alpha) = \frac{\mu}{\alpha+\mu} \sum_{j\in C} m_j. \tag{1.22}$$

Now, since $F$ satisfies (FE$_{ij}$) over $S$, we have in particular that

$$\alpha \phi_{i0}(\alpha) = \sum_{j\in C} \phi_{ij}(\alpha) q_{j0}, \qquad i \in C, \tag{1.23}$$

and so, again using (1.21), we get

$$\sum_{i\in C} m_i \alpha \phi_{i0}(\alpha) = \frac{1}{\alpha+\mu} \sum_{i\in C} m_i q_{i0}.$$

This expression combines with (1.22) and (1.17) to give $\sum_{i\in C} m_i z_i(\alpha) = 0$, which is a contradiction because $z$ is not identically 0. We deduce that $m$ cannot be $\mu$-invariant for $F$. Moreover, we must have

$$\sum_{i\in C} m_i (\alpha+\mu) \phi_{ij}(\alpha) < m_j \tag{1.24}$$

for at least one $j \in C$, and hence, from (1.23) and (1.17),

$$\sum_{i\in C} m_i (\alpha+\mu) \phi_{i0}(\alpha) < \frac{1}{\alpha} \sum_{i\in C} m_i q_{i0} = \frac{\mu}{\alpha} \sum_{i\in C} m_i.$$

Thus, $d_0(\alpha) > 0$ and $d_j(\alpha) > 0$ for at least one $j \in C$.

Next we shall show that $\Psi(\cdot) = (\psi_{ij}(\cdot),\ i,j \in S)$, given by (1.18), is the resolvent of an honest $Q$-process $P$ and that $m$ is a $\mu$-invariant measure for $P$. Clearly $\psi_{ij}(\alpha) \geq 0$ for all $i,j \in S$. Since $m$ is finite, we have, from the definition of $d$, that

$$\alpha \sum_{j\in S} d_j(\alpha) = (\alpha+\mu) \sum_{j\in C} m_j z_j(\alpha)$$

and so $\sum_{j\in S} \alpha \psi_{ij}(\alpha) = 1$ for all $i \in S$. In order to prove that $\Psi$ is the resolvent of a standard transition function $P$, we need only show that $\Psi$ satisfies the resolvent equation (1.13); see Theorem 1 of (Reuter, 1967). We shall use the following identities:

$$z_i(\alpha) - z_i(\beta) + (\alpha-\beta) \sum_{k\in C} \phi_{ik}(\alpha) z_k(\beta) = 0, \qquad i \in C, \tag{1.25}$$

$$d_i(\alpha) - d_i(\beta) + (\alpha-\beta) \sum_{k\in C} d_k(\alpha) \phi_{ki}(\beta) = 0, \qquad i \in C, \tag{1.26}$$

$$\alpha d_0(\alpha) - \beta d_0(\beta) + (\alpha - \beta) \sum_{k \in C} d_k(\alpha) \beta \phi_{k0}(\beta) = 0 \tag{1.27}$$

and

$$(\alpha+\mu) \sum_{i \in C} m_i z_i(\alpha) - (\beta+\mu) \sum_{i \in C} m_i z_i(\beta) = (\alpha-\beta) \sum_{i \in C} d_i(\alpha) z_i(\beta). \tag{1.28}$$

The first three of these can be verified directly using the fact that $\Phi$ satisfies the resolvent equation and that $z_0(\alpha) = 0$. The fourth identity follows from the first on multiplying by $m_i$ and summing over $i$. Using (1.25) – (1.28), together with the resolvent equation for $\Phi$, it is easy to prove that $\Psi$ satisfies its own resolvent equation.

Next we need to verify that $P$ is indeed a $Q$-process, that is $p'_{ij}(0+) = q_{ij}$ for all $i, j \in S$. We shall use a remark of Reuter on Page 83 of (Reuter, 1959) (see also Theorem 3.1 of (Feller, 1957)): if one is given a standard transition function $P$, then it is a $Q$-process if and only if the backward equations hold, equivalently,

$$\alpha \psi_{ij}(\alpha) = \delta_{ij} + \sum_{k \in S} q_{ik} \psi_{kj}(\alpha), \qquad i, j \in S.$$

But, this follows almost immediately from the identity

$$\sum_{k \in S} q_{ik} z_k(\alpha) = \alpha z_i(\alpha), \qquad i \in S,$$

which can be deduced from the backward equations for $\Phi$.

We have shown that $\Psi$ is the resolvent of a $Q$-process $P$. To show that $m$ is a $\mu$-invariant measure for $P$, we again use the definition of $d$: it is elementary to check that

$$\sum_{i \in S} m_i (\alpha + \mu) \alpha \psi_{ij}(\alpha) = m_j, \qquad j \in S.$$

We have already seen that $P$ is honest and so it remains only to show that $P$ satisfies (FE$_{i0}$) for $i \in C$. But, since $z$ is not identically $0$, this happens when and only when

$$\sum_{i \in C} d_i(\alpha) q_{i0} = \alpha d_0(\alpha),$$

because it is easily verified that

$$\sum_{k \in C} \psi_{ik}(\alpha) q_{k0} = \alpha \psi_{i0}(\alpha) + C_\alpha z_i(\alpha) \left( \sum_{k \in C} d_k(\alpha) q_{k0} - \alpha d_0(\alpha) \right),$$

where

$$C_\alpha^{-1} = (\alpha + \mu) \sum_{k \in C} m_k z_k(\alpha).$$

On substituting for $d$, we find that

$$\alpha d_0(\alpha) - \sum_{i \in C} d_i(\alpha) q_{i0} = \mu \sum_{k \in C} m_k - \sum_{k \in C} m_k q_{k0} = 0,$$

and so the result follows.

*Remarks*. (1) The final part of the theorem states that the process $P$ we have identified satisfies ($\text{FE}_{i0}$) for $i \in C$. The remaining forward equations do not necessarily hold. By Theorem 3.1 of (Nair and Pollett, 1993), this happens when and only when $m$ is $\mu$-invariant for $Q$ (rather than merely $\mu$-subinvariant). Indeed, a simple calculation shows that, for all $j \in C$,

$$\sum_{k \in C} \psi_{ik}(\alpha) q_{kj} = \alpha \psi_{ij}(\alpha) - \delta_{ij} + C_\alpha z_i(\alpha) \left( \sum_{k \in C} d_k(\alpha) q_{kj} - \alpha d_j(\alpha) \right),$$

and, for the given $P$,

$$\alpha d_j(\alpha) - \sum_{i \in C} d_i(\alpha) q_{ij} = -\mu m_j - \sum_{i \in C} m_i q_{ij} \ (\geq 0), \qquad j \in C,$$

this later quantity measuring the "$\mu$-invariance deficit" of $m$ for $Q$.

(2) A straightforward calculation shows that the given $\Psi$ satisfies

$$\sum_{i \in C} m_i (\alpha + \mu) \psi_{i0}(\alpha) = \mu \sum_{i \in C} m_i,$$

and hence $m$ satisfies a set of "residual equations" for $P$, namely

$$\sum_{i \in C} m_i p_{i0}(t) = (1 - e^{-\mu t}) \sum_{i \in C} m_i, \qquad t > 0, \tag{1.29}$$

which can be regarded as a "process counterpart" to (1.17). (Since $P$ is honest, (1.29) follows more directly on summing (1.8) over $j \in C$.)

## 4. NECESSARY CONDITIONS

It would be tempting to conjecture that condition (1.17) is *necessary* for the existence of a $Q$-process for which the given measure is $\mu$-invariant. However, while this is *not* the case, condition (1.17) turns out to be necessary when extra conditions are imposed.

Let $P$ be a $Q$-process with $C$ being an irreducible class (the conditions we have imposed on $Q$ ensure that 0 is an absorbing state which is accessible from $C$) and suppose that $m = (m_j, \ j \in C)$ is a finite $\mu$-invariant measure for $P$. Of necessity, $m$ will be $\mu$-subinvariant for $Q$, but does (1.17) necessarily hold? Under the conditions we have imposed, the *forward integral inequalities* are satisfied (Reuter, 1957); in particular,

$$p_{i0}(t) \geq \sum_{k\in C} \int_0^t p_{ik}(s) q_{k0}\, ds, \qquad i \in C. \tag{1.30}$$

On multiplying by $m_i$ and summing over $i \in C$, we find that

$$(1 - e^{-\mu t}) \sum_{k\in C} m_k q_{k0} \leq \mu \sum_{i\in C} m_i p_{i0}(t)\ (< \infty). \tag{1.31}$$

If we divide by $\mu$ and let $t \to 0$, we may use dominated convergence to deduce that

$$\mu \sum_{i\in C} m_i a_i \geq \sum_{i\in C} m_i q_{i0},$$

where $a_i$ (the probability of absorption starting in state $i$) is given by $a_i = \lim_{t\to\infty} p_{i0}(t)$. Thus, if $a_i$ is strictly less than 1 for some (and then all) $i \in C$, (1.17) cannot hold.

If we were to assume that $P$ satisfies ($\mathrm{FE}_{i0}$) over $C$, then we would have equality in (1.30) and (1.31), and so

$$\mu \sum_{i\in C} m_i a_i = \sum_{i\in C} m_i q_{i0}.$$

If instead $P$ were assumed to be *honest*, then we would have $a_i = 1$ for all $i \in C$. This can be seen as follows. Since $m$ is a $\mu$-invariant measure for $P$, we have, in particular, that $m_i p_{ij}(t) \leq e^{-\mu t} m_j$ for $i, j \in C$, and so

$$1 - p_{i0}(t) = \sum_{j\in C} p_{ij}(t) \leq e^{-\mu t} \frac{1}{m_i} \sum_{j\in C} m_j, \qquad i \in C.$$

Since $m$ is finite and $\mu > 0$, $\lim_{t\to\infty}(1 - p_{i0}(t)) = 0$, and hence $a_i = 1$ for all $i \in C$. Thus, if $P$ were honest, we would have

$$\mu \sum_{i\in C} m_i \geq \sum_{i\in C} m_i q_{i0}.$$

Neither the honesty of $P$, nor an assumption that $P$ satisfies ($\mathrm{FE}_{i0}$) over $C$, is enough on its own to establish (1.17); it is possible to construct examples of $Q$-processes which illustrate this. But, these conditions *together* imply (1.17).

We have therefore proved the following variant of Theorem 1:

**Theorem 2** *Let $\mu > 0$ and suppose that $Q$ admits a finite $\mu$-subinvariant measure. Then, there exists an honest $Q$-process $P$ satisfying* ($\mathrm{FE}_{i0}$) *over $C$ for which $m$ is $\mu$-invariant if and only if (1.17) holds. The resolvent of one such $Q$-process is given by (1.18).*

Next we shall examine the question of *uniqueness* under the assumption that $Q$ is a single-exit $Q$-matrix. This was considered briefly in Section 5 of (Nair and Pollett, 1993) under a condition weaker than (1.17). If $Q$ is single exit and $P$ is an arbitrary $Q$-process, then (Reuter, 1959) either $P$ is the minimal $Q$-process or otherwise its resolvent $\Psi(\cdot) = (\psi_{ij}(\cdot),\ i, j \in S)$ must be of the form

$$\psi_{ij}(\alpha) = \phi_{ij}(\alpha) + z_i(\alpha) y_j(\alpha), \qquad i, j \in S, \tag{1.32}$$

where $y(\alpha) = (y_j(\alpha),\ j \in S)$ is given by

$$y_j(\alpha) = \frac{\eta_j(\alpha)}{c + \sum_{k \in S} \alpha \eta_k(\alpha)}, \qquad j \in S, \tag{1.33}$$

$c$ is a non-negative constant, and $\eta(\alpha) = (\eta_j(\alpha),\ j \in S)$ is a non-negative vector which satisfies

$$\sum_{k \in S} \eta_k(\alpha) < \infty, \tag{1.34}$$

$$\eta_j(\alpha) - \eta_j(\beta) + (\alpha - \beta) \sum_{k \in S} \eta_k(\alpha) \phi_{kj}(\beta) = 0, \quad j \in S. \tag{1.35}$$

Furthermore, $\Psi$ is honest if and only if $c = 0$. Since we have assumed that $0$ is an absorbing state, $z_0(\alpha) = 0$ and so (1.35) can be written

$$\eta_j(\alpha) - \eta_j(\beta) + (\alpha - \beta) \sum_{k \in C} \eta_k(\alpha) \phi_{kj}(\beta) = 0, \quad j \in C, \tag{1.36}$$

$$\alpha \eta_0(\alpha) - \beta \eta_0(\beta) + (\alpha - \beta) \sum_{k \in C} \eta_k(\alpha) \beta \phi_{k0}(\beta) = 0. \tag{1.37}$$

Once $\eta$ is determined, a family of $Q$-processes, exactly one of which is honest, is obtained by varying $c$ in the range $0 \leq c < \infty$. Thus, the problem of identifying those $Q$-processes which satisfy a specified criterion, in our case, that a given measure is $\mu$-invariant on $C$, amounts to determining which choices of $\eta$ and $c$ are admissible; the procedure is purely arithmetical.

**Theorem 3** *Suppose that $Q$ is single exit and suppose that, for a given $\mu > 0$, $Q$ admits a finite $\mu$-subinvariant measure. Then, there exists an honest $Q$-process $P$ satisfying* (FE$_{i0}$) *over $C$ for which $m$ is $\mu$-invariant if and only if (1.17) holds. It is the unique honest $Q$-process for which $m$ is $\mu$-invariant and its resolvent is given by (1.18).*

*Proof*. In view of Theorem 2, we only need to establish uniqueness. If the minimal $Q$-process $F$ is honest, then it is the unique $Q$-process, and, as we have already observed, (1.17) is necessary and sufficient for $m$ to be $\mu$-invariant for $F$.

Suppose then that $F$ is dishonest, so that $z$ is not identically 0. We will prove that if there *is* an honest $Q$-process $P$ for which $m$ is $\mu$-invariant, then its resolvent must necessarily be given by (1.18).

Let $d$ be given by (1.19) and (1.20). Since $m$ is $\mu$-invariant for $P$, multiplying (1.32) by $(\alpha+\mu)m_i$ and summing over $i \in C$ gives

$$m_j = \sum_{i\in C} m_i(\alpha+\mu)\phi_{ij}(\alpha) + (\alpha+\mu)y_j(\alpha)\sum_{i\in C} m_i z_i(\alpha),$$

for all $j \in C$. Since $P$ is honest, we must set $c = 0$ and so in view of (1.33) we require

$$\frac{\eta_j(\alpha)}{\sum_{k\in S}\alpha\eta_k(\alpha)} = \frac{d_j(\alpha)}{(\alpha+\mu)\sum_{i\in C} m_i z_i(\alpha)}, \quad j \in C. \tag{1.38}$$

Notice that $d_j(\alpha) > 0$ for at least one $j \in C$: since $m$ is $\mu$-invariant for $P$, $m$ cannot be $\mu$-invariant for $F$, and so (1.24) holds for at least one $j \in C$. Furthermore, by the definition of $d$, we have that

$$\alpha \sum_{j\in S} d_j(\alpha) = (\alpha+\mu)\sum_{j\in C} m_j z_j(\alpha) < \infty, \tag{1.39}$$

which is consistent with (1.34). From (1.38) we see that $\eta_j(\alpha) = K(\alpha)d_j(\alpha)$, at least for $j \in C$, where $K$ is some positive scalar function. Using the identity (1.26), together with the fact that $\eta$ must satisfy (1.36), we find, on substituting $\eta_j(\alpha) = K(\alpha)d_j(\alpha)$ in (1.36), that $(K(\alpha) - K(\beta))d_j(\beta) = 0$. Hence, $K$ must be constant, because $d_j(\beta) > 0$ for at least one $j \in C$. Now, using (1.38) again, we see that $K$ must satisfy

$$K\left((\alpha+\mu)\sum_{k\in C} m_k z_k(\alpha) - \alpha\sum_{k\in C} d_k(\alpha)\right) = \alpha\eta_0(\alpha),$$

or equivalently, by (1.39),

$$K\alpha d_0(\alpha) = \alpha\eta_0(\alpha). \tag{1.40}$$

It is clear from (1.27) that $\eta_0$ satisfies (1.37) no matter what the value of $K$. It is also clear that there is no loss of generality in setting $K = 1$, for this is equivalent to replacing $c$ in (1.38) by a different constant $c/K$. Hence $\eta_j = d_j$ for $j \in C$, and, from (1.40), $\eta_0 = d_0$.

We have proved that if $Q$ is single exit and $P$ is an honest $Q$-process with $\mu$-invariant measure $m$, then its resolvent must be given by (1.18).

## Acknowledgments

I would like to thank the organizers for their invitation to speak at the workshop, and for their kind hospitality during my visit to Changsha.

## References

Chen, A. and Zhang, H. (1987). Criterion for the existence of reversible $Q$-processes. *Acta Math. Sinica*, 3:133–142.

Feller, W. (1940). On the integro-differential equations of purely discontinuous Markoff processes. *Trans. Amer. Math. Soc.*, 48:488–515.

Feller, W. (1957). On boundaries and lateral conditions for the Kolmogoroff differential equations. *Ann. Math.*, 65:527–570.

Hou, C. (1974). The criterion for uniqueness of a $Q$-process. *Sci. Sinica*, XVII: 141–159.

Hou, C. and Chen, M. (1980). Markov processes and field theory. *Kexue Tongbao*, 25:807–811.

Kingman, J. (1963). The exponential decay of Markov transition probabilities. *Proc. London Math. Soc.*, 13:337–358.

Nair, M. and Pollett, P. (1993). On the relationship between $\mu$-invariant measures and quasistationary distributions for continuous-time Markov chains. *Adv. Appl. Probab.*, 25:82–102.

Pollett, P. (1986). On the equivalence of $\mu$-invariant measures for the minimal process and its $q$-matrix. *Stochastic Process. Appl.*, 22:203–221.

Pollett, P. (1988). Reversibility, invariance and $\mu$-invariance. *Adv. Appl. Probab.*, 20:600–621.

Pollett, P. (1991a). Invariant measures for $Q$-processes when $Q$ is not regular. *Adv. Appl. Probab.*, 23:277–292.

Pollett, P. (1991b). On the construction problem for single-exit Markov chains. *Bull. Austral. Math. Soc.*, 43:439–450.

Pollett, P. (1994). On the identification of continuous-time Markov chains with a given invariant measure. *J. Appl. Probab.*, 31:897–910.

Pollett, P. and Vere-Jones, D. (1992). A note on evanescent processes. *Austral. J. Statist.*, 34:531–536.

Reuter, G. (1957). Denumerable Markov processes and the associated contraction semigroups on $l$. *Acta Math.*, 97:1–46.

Reuter, G. (1959). Denumerable Markov processes (II). *J. London Math. Soc.*, 34:81–91.

Reuter, G. (1962). Denumerable Markov processes (III). *J. London Math. Soc.*, 37:63–73.

Reuter, G. (1967). Note on resolvents of denumerable submarkovian processes. *Z. Wahrscheinlichkeitstheorie*, 9:16–19.

Rogers, L. and Williams, D. (1986). Construction and approximation of transition matrix functions. In Kendall, D., Kingman, J., and Williams, D., editors, *Analytic and Geometric Stochastics, Papers in honour of G.E.H. Reuter*, pages 133–160. Applied Probability Trust.

Tweedie, R. (1974). Some ergodic properties of the Feller minimal process. *Quart. J. Math. Oxford*, 25:485–495.

van Doorn, E. (1991). Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. Appl. Probab.*, 23:683–700.

Vere-Jones, D. (1967). Ergodic properties on non-negative matrices I. *Pacific J. Math.*, 22:361–386.

Williams, D. (1964). On the construction problem for Markov chains. *Z. Wahrscheinlichkeitstheorie*, 3:227–246.

Zhang, H., Lin, X., and Hou, Z. (1998a). Invariant distribution of $Q$-process (I). Submitted for publication.

Zhang, H., Lin, X., and Hou, Z. (1998b). Invariant distribution of $Q$-process (II). Submitted for publication.