MODELLING RANDOM FLUCTUATIONS IN A BISTABLE TELECOMMUNICATIONS NETWORK

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Abstract. In this paper I shall consider a model for the simplest kind of dynamic routeing in a circuit-switched telecommunications network, namely Random Alternative Routeing: if a call cannot be carried on a first-choice route, then a second-choice route is chosen at random from a fixed set of alternatives. This kind of routeing can give rise to several modes of behaviour. For example, the simple model I shall consider can exhibit bistability; the system fluctuates between a low-blocking state, where calls are accepted readily, and a high-blocking state, where the likelihood of a call being accepted can be quite low. I shall describe a method which allows one to study the stability of the two states. In particular, the method allows one to estimate the time for which these states persist.

1. Introduction

Recently, Gibbens et al. (1990) introduced a simple model which helps to explain why circuit-switched telecommunications networks with Random Alternative Routeing can exhibit bistable behaviour. Such bistability can have serious implications for the performance of the network, for, in the high-blocking state, a situation can persist where large numbers of calls use alternative "second-choice" routes, which generally demand greater link occupancy than do "first-choice" routes and, thus, new calls are likely to be blocked frequently. The persistence of the high-blocking state is brought about because, even when calls are accepted, they are allotted "first-choice" routes rather rarely. It is of interest, therefore, to determine the time it takes for the system to relax to the low-blocking state, where new calls are accepted more readily, and then to determine the time for which the low-blocking state persists. In this paper I shall describe a method by which one can model the random fluctuations of the system about its various states, either stable or unstable. For example, the method allows one to show that, as the number of links becomes large, the distribution of the time it takes to leave a region containing a stable equilibrium is, asymptotically, negative exponential.

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2. The model : a symmetric fully-connected network

There are N nodes connected to one another and, thus, a total of $K = \binom{N}{2}$ links (circuit groups). The links are assumed to have the same number of circuits, C, and calls between any two given nodes, a and b, arrive according to a Poisson process with rate $\nu > 0$; the arrival streams are assumed to be independent. If a call is offered to the link connecting a and b, and there is at least one free circuit on that link, then the call is connected and is held for a negative exponentially distributed period with mean 1. If there are no free circuits on the direct link, a third node, c, is chosen at random from the remaining nodes and an attempt is made to connect the call on the route via c. If there is spare capacity on each of the two links comprising the alternative route, the call is connected and holds one circuit from each link for a period that is negative exponentially distributed with mean 1. The call is blocked, and then lost, if it cannot be accommodated on the alternative route. Call lengths are assumed to be mutually independent, and independent of the arrival process.

Although the usual model of this network is a finite-state irreducible Markov chain, its state description is rather complicated and its equilibrium behaviour cannot be analysed simply. For this reason, Gibbens et al. (1990) proposed a simplified model which does not respect the graph structure of the network, but whose behaviour for large N is a good approximation to that of the original model. The simplified description of the network, which I shall refer to as the GHK model, differs in two ways. In cases when a call cannot be connected on a direct link, two *links* are chosen *at random* from the remaining $K-1$ links. If there is spare capacity on each of these links, one circuit is held on each link, but, now, the two holding times are *independent* negative exponentially distributed random variables with mean 1. Thus, in contrast to the original model, the two circuits are released at *different* times (with probability 1).

If $n_j^{(K)}(t)$ is the number of links with j circuits in use at time t, for a network with K links, then, under the above assumptions, $(n^{(K)}(t),\ t \geq 0)$, where $n^{(K)} = (n_0^{(K)}, n_1^{(K)}, \ldots, n_C^{(K)})$, is a continuous-time Markov chain which takes values in

$$S^{(K)} = \left\{ n \in \{0, 1, \ldots\}^{C+1} \ :\ \sum_{i=0}^{C} n_i = K \right\}$$

and which has transition rates, $Q^{(K)} = (q^{(K)}(n, n'),\ n, n' \in S^{(K)})$, given by

$$\begin{aligned}
q^{(K)}(n, n + e_{j+1} - e_j) &= \nu n_j, \qquad 0 \leq j \leq C-1, \\
q^{(K)}(n, n + e_{j-1} - e_j) &= j n_j, \qquad 1 \leq j \leq C, \\
q^{(K)}(n, n + e_{i+1} - e_i + e_{j+1} - e_j) &= \nu n_C \frac{n_i n_j}{\binom{K}{2}}, \qquad i > j, 0 \leq i, j \leq C-1, \\
q^{(K)}(n, n + 2(e_{j+1} - e_j)) &= \nu n_C \frac{\binom{n_j}{2}}{\binom{K}{2}}, \qquad 0 \leq j \leq C-1,
\end{aligned}$$

where e_i is the unit vector with 1 as its i^{th} entry.

Gibbens et al. (1990) proved a functional law of large numbers for the simplified model, and this has recently been shown to be valid for the original model, subject to certain natural constraints on the initial state of the network (See Crametz and Hunt (1990)). In particular, they considered the behaviour of $X^{(K)} = (X_1^{(K)}, X_2^{(K)}, \ldots, X_C^{(K)})$, where

$$X_j^{(K)}(t) = \frac{n_j^{(K)}(t)}{K}$$

is the *proportion* of links with j circuits in use at time t. The process $(X^{(K)}(t),\ t \geq 0)$ is itself a Markov chain, but one which takes values in a lattice contained in the simplex

$$E = \left\{ x \in [0,\infty)^{C+1} \ : \ \sum_{i=0}^{C} x_i = 1 \right\}. \tag{2.1}$$

Gibbens et al. showed that if, in the limit as $K \to \infty$, $X^{(K)}(0) \Rightarrow x_0$ in E, then $X^{(K)}(\cdot) \Rightarrow X(\cdot, x_0)$ in $D_E[0,\infty)$, where $(X(t,x_0),\ t \geq 0)$ is a deterministic process with initial point $X(0,x_0) = x_0$; here $\Rightarrow$ denotes weak convergence and $D_E[0,\infty)$ is the space of all sample paths on $[0,\infty)$. By studying the behaviour of $X(t,x_0)$ in the limit as $t \to \infty$, they were able to demonstrate the possibility of bistable behaviour for C large enough and for a narrow range of values of the ratio ν/C.

The law of large numbers establishes that, when K is large, the sample behaviour of the model can be approximated by $X(\cdot, x_0)$. However, it does not provide information concerning the random fluctuations about this deterministic path. For this reason, I shall propose an analogous functional central limit theorem that establishes a diffusion approximation for $X^{(K)}(\cdot)$ which is valid over finite intervals of time. This will be made possible by observing that the GHK model is *asymptotically density dependent*, in the sense of Pollett (1990). I shall first recall this notion and then state two functional limit laws for asymptotically density dependent Markov chains which are appropriate for analysing the GHK model.

3. Asymptotic density dependence

Let $\{n^{(K)}(\cdot)\}$ be a family of continuous-time Markov chains, indexed by $K > 0$, and suppose that $n^{(K)}(\cdot)$ takes values in $S^{(K)}$, a subset of $\mathbb{Z}^J$, and has transition rates $Q^{(K)} = (q^{(K)}(n,n'),\ n, n' \in S^{(K)})$.

Definition. Suppose that there exists an open set $E \subseteq \mathbb{R}^J$ and a family, $\{f^{(K)},\ K > 0\}$, of continuous functions, with $f^{(K)} : E \times \mathbb{Z}^J \to \mathbb{R}$, such that

$$q^{(K)}(n, n+l) = K f^{(K)}\left(\frac{n}{K}, l\right), \qquad l \neq 0, \tag{3.1}$$

and $\sum_l l f^{(K)}(x,l)$ converges for all $x \in E$. Then the family of Markov chains is asymptotically density dependent if, in addition, there exists a function, $F : E \to \mathbb{R}^J$, such that $\{F^{(K)}\}$, given by

$$F^{(K)}(x) = \sum_l l f^{(K)}(x,l), \qquad x \in E,$$

converges to F on E.

This definition of density dependence differs from that introduced by Kurtz (1970). His definition requires only that there exists a continuous function, $f : \mathbb{R}^J \times \mathbb{Z}^J \to \mathbb{R}$, such that

$$q^{(K)}(n, n+l) = Kf\left(\frac{n}{K}, l\right), \qquad l \neq 0,$$

and (implicitly) $\sum_l lf(x,l) < \infty$ for all x. Thus, an asymptotically density dependent family of Markov chains is density dependent if $f^{(K)}$ (and hence $F^{(K)}$) does not depend on K. Roughly speaking, a family is density dependent if the transition rates of the corresponding "density process", $X^{(K)}(\cdot)$, defined by

$$X^{(K)}(t) = \frac{n^{(K)}(t)}{K}, \qquad t \geq 0,$$

depend on the present state, n, only through the density n/K; an *asymptotically* density dependent family is one which exhibits this property in the limit as $K \to \infty$. Thus, there is a natural way to associate with this process a density dependent deterministic process which, for large K, is "tracked" by the process. Indeed, a straightforward formal argument based on the Kolmogorov forward differential equations for the state probabilities, shows that, for large K,

$$\frac{d}{dt}\mathbb{E}X^{(K)}(t) \simeq \mathbb{E}F^{(K)}(X^{(K)}(t)), \qquad t > 0.$$

Thus one might expect this deterministic process, call it $X(\cdot)$, to satisfy

$$\frac{d}{dt}X(t) = F(X(t)), \qquad t > 0.$$

The following "law of large numbers" establishes that, under appropriate conditions, the density process *does* track a deterministic process; see Pollett (1990) for details.

Theorem 3.1. *Suppose that F is Lipschitz continuous on E and that, for all $K > 0$,*

$$\sup_{x \in E} \sum_l |l| f^{(K)}(x,l) < \infty, \tag{3.2}$$

$$\lim_{\delta \to \infty} \sup_{x \in E} \sum_{l : |l| > \delta} |l| f^{(K)}(x,l) = 0 \tag{3.3}$$

and

$$\lim_{K \to \infty} \sup_{x \in E} |F^{(K)}(x) - F(x)| = 0. \tag{3.4}$$

Then, if

$$\lim_{K \to \infty} X^{(K)}(0) = x_0, \tag{3.5}$$

we have that

$$\lim_{K\to\infty} \Pr\left(\sup_{s\le t}\left|X^{(K)}(s) - X(s,x_0)\right| > \epsilon\right) = 0, \qquad 0 \le s \le t,$$

for all $\epsilon > 0$, *and for every trajectory*, $X(\cdot, x_0)$, *satisfying*

$$\begin{gathered} X(0,x_0) = x_0, \\ X(s,x_0) \in E, \qquad 0 \le s \le t, \\ \frac{\partial}{\partial s} X(s,x_0) = F(X(s,x_0)). \end{gathered}$$

Conditions (3.2) and (3.3), together with the Lipschitz condition on F, are usually satisfied in most practical situations. For example, (3.2) and (3.3) hold if, for each l, $f^{(K)}(\cdot, l)$ is bounded (on E) and if there are only a finite number of transitions out of any state, n. Condition (3.5) is important. It stipulates that the density process should begin close to the initial value, x_0, of the deterministic trajectory. The conclusion of the theorem is, then, that the density process converges (uniformly in probability) over any finite time interval, to the deterministic path.

The following "central limit law" establishes that, for large K, the fluctuations about the deterministic path follow a diffusion, provided that certain "second-order" conditions are satisfied; again see Pollett (1990) for details.

Theorem 3.2. *Suppose that* $\{F^{(K)}\}$ *converges uniformly to* F *and that* F *is bounded and Lipschitz continuous on* E. *Suppose also that the family* $\{G^{(K)}\}$, *where* $G^{(K)}(x)$ *is a* $J \times J$ *matrix with elements*

$$g_{ij}^{(K)}(x) = \sum_l l_i l_j f^{(K)}(x,l), \qquad x \in E,$$

converges uniformly to G, *where* G *is bounded and uniformly continuous on* E.

If, in addition,

$$\sup_{x\in E} \sum_l |l|^2 f^{(K)}(x,l) < \infty, \tag{3.6}$$

$$\lim_{\delta\to\infty} \sup_{x\in E} \sum_{l:|l|>\delta} |l|^2 f^{(K)}(x,l) = 0, \tag{3.7}$$

for all $K > 0$, *and*

$$\lim_{K\to\infty} \sup_{x\in E} \sqrt{K}|F^{(K)}(x) - F(x)| = 0, \tag{3.8}$$

where now F *is assumed to have uniformly continuous first partial derivatives, then, provided*

$$\lim_{K\to\infty} \sqrt{K}\left(X^{(K)}(0) - x_0\right) = z, \tag{3.9}$$

the family of processes $\{Z^{(K)}(\cdot)\}$, *defined by*

$$Z^{(K)}(s) = \sqrt{K}\left(X^{(K)}(s) - X(s,x_0)\right), \qquad 0 \le s \le t,$$

converges weakly in $D_E[0,t]$ *to a diffusion,* $Z(\cdot)$, *with initial value* $Z(0) = z$ *and with characteristic function,* $\psi = \psi(s,\theta)$, *which satisfies*

$$\frac{\partial\psi}{\partial s}(s,\theta) = -\frac{1}{2}\sum_{j,k}\theta_j g_{jk}(X(s,x_0))\theta_k\psi(s,\theta) + \sum_{j,k}\theta_j\frac{\partial F_j}{\partial x_k}(X(s,x_0))\frac{\partial\psi}{\partial\theta_k}(s,\theta). \tag{3.10}$$

Again, the technical conditions, (3.6) and (3.7), hold if $f^{(K)}(\cdot,l)$ is bounded for each l and there are finitely many possible transitions out of each state. Condition (3.8) strengthens (3.4) to ensure that $\{F^{(K)}\}$ converges to F at the correct rate, while Condition (3.9) provides the initial value of the diffusion.

4. A functional central limit theorem for the GHK model

The GHK model is clearly asymptotically density dependent, for one can define E by (2.1) and $f^{(K)} : E \times \mathbb{Z}^{(C+1)} \to \mathbb{R}$, $K \ge 1$, by

$$\begin{aligned}
f^{(K)}(x, e_{j+1} - e_j) &= \nu x_j, \qquad 0 \le j \le C-1,\\
f^{(K)}(x, e_{j-1} - e_j) &= j x_j, \qquad 1 \le j \le C,\\
f^{(K)}(x, e_{i+1} - e_i + e_{j+1} - e_j) &= 2\nu\left(\frac{K}{K-1}\right)x_C x_i x_j, \qquad i > j, 0 \le i,j \le C-1,\\
f^{(K)}(x, 2(e_{j+1} - e_j)) &= \nu\left(\frac{K}{K-1}\right)x_C x_j\left(x_j - \frac{1}{K}\right), \qquad 0 \le j \le C-1,
\end{aligned}$$

so that (3.1) is satisfied. It is then clear that, as $K \to \infty$, $f^{(K)}$ converges (uniformly on E) to f, given by

$$\begin{aligned}
f(x, e_{j+1} - e_j) &= \nu x_j, \qquad 0 \le j \le C-1,\\
f(x, e_{j-1} - e_j) &= j x_j, \qquad 1 \le j \le C,\\
f(x, e_{i+1} - e_i + e_{j+1} - e_j) &= 2\nu x_C x_i x_j, \qquad i > j, 0 \le i,j \le C-1,\\
f(x, 2(e_{j+1} - e_j)) &= \nu x_C x_j^2, \qquad 0 \le j \le C-1,
\end{aligned}$$

and so the corresponding sequence $\{F^{(K)}\}$, defined by $F^{(K)}(x) = \sum_l l f^{(K)}(x,l)$, $x \in E$, converges (uniformly on E) to F, given by $F(x) = \sum_l l f(x,l)$, $x \in E$. On evaluating this latter summation, one finds that

$$F(x) = (H^T + \lambda(x)R^T)x,$$

where $\lambda(x) = 2\nu x_C(1 - x_C)$, and H and R are $(C+1) \times (C+1)$ matrices, given by

$$H = \begin{pmatrix} -\nu & \nu & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & -(\nu+1) & \nu & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & -(\nu+2) & \nu & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & C-1 & -(\nu+C-1) & \nu \\ 0 & 0 & 0 & 0 & \dots & 0 & -C & C \end{pmatrix}$$

and

$$R = \begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Thus, the appropriate deterministic model to consider is

$$\begin{aligned} \frac{d}{dt}X_0(t) &= X_1(t) - (\nu + \lambda(t))X_0(t), \\ \frac{d}{dt}X_j(t) &= (\nu + \lambda(t))X_{j-1}(t) - (\nu + \lambda(t) + j)X_j(t) + (j+1)X_{j+1}(t), \\ &\qquad\qquad 1 \le j \le C-1, \\ \frac{d}{dt}X_C(t) &= (\nu + \lambda(t))X_{C-1}(t) - CX_C(t), \end{aligned} \tag{4.1}$$

where $\lambda(t) = \lambda(X(t))$. We might have expected this kind of law of motion to govern the limit proportions, for notice that H is the transition matrix of an Erlang loss system with C circuits and with Poisson traffic offered at rate ν, and so if λ were identically zero, (4.1) would comprise the forward equations for the state probabilities of such a system. As it is, $\lambda(t)$ gives the additional arrival rate at time t due to overflowing calls. As Gibbens et al. point out, (4.1) admits a unique solution, $X(\cdot, x_0)$, for each given initial point, $X(0, x_0) = x_0$; this follows from the fact that F is Lipschitz continuous on E. Now, (3.2) and (3.3) are trivially satisfied and (3.4) holds because

$$|F_j^{(K)}(x) - F_j(x)| = \frac{1}{K-1}(2\nu - \lambda(x))\Delta x_j, \tag{4.2}$$

where

$$\Delta x_j = \begin{cases} -x_0, & j = 0, \\ x_{j-1} - x_j, & 1 \le j \le C-1, \\ x_{C-1}, & j = C. \end{cases}$$

Thus, using Theorem 3.1, we have the following version of the law of large numbers for the GHK model:

Theorem 4.1. *In the GHK model, let $X_j^{(K)}(t)$ be the proportion of links with j circuits in use at time t and define $(X^{(K)}(t),\ t \geq 0)$ by $X^{(K)} = (X_1^{(K)}, X_2^{(K)}, \ldots, X_C^{(K)})$. Then, if*

$$\lim_{K\to\infty} X^{(K)}(0) = x_0,$$

we have that

$$\lim_{K\to\infty} \Pr\left(\sup_{s\leq t}\left|X^{(K)}(s) - X(s,x_0)\right| > \epsilon\right) = 0, \qquad 0 \leq s \leq t,$$

for all $\epsilon > 0$, where $X(\cdot, x_0)$ is the unique solution to (4.1) such that $X(0,x_0) = x_0$.

The theorem allows us to conclude, for example, that $\{X^{(K)}(s)\}$ converges in probability to $X(s,x_0)$ and, since for each s, $X^{(K)}(s)$ is uniformly bounded, dominated convergence implies that

$$\lim_{K\to\infty} \mathbb{E}X^{(K)}(s) = X(s,x_0)$$

on all finite time intervals.

The additional conditions of Theorem 3.2 are also satisfied, and so one can establish an analogous central limit law. In particular, F is bounded and, since, for each l, $f^{(K)}(\cdot, l)$ converges uniformly on E to $f(\cdot, l)$, we have that $\{F^{(K)}\}$ converges uniformly to F. The second-order technical conditions, (3.6) and (3.7), are trivially satisfied, as is (3.8), by virtue of (4.2). A suitable sequence of covariance matrices, $\{G^{(K)}\}$, can be constructed from $\{f^{(K)}\}$ and it is easy to see that this sequence converges uniformly to the matrix G with elements

$$g_{ij}(x) = \sum_l l_i l_j f(x,l), \qquad x \in E.$$

Although the precise arithmetical evaluation of G is tedious, it is clear that G is bounded and uniformly continuous on E. This follows from the definition of G and the fact that, for each l, $f(\cdot, l)$ is bounded and uniformly continuous on E. Thus, provided (3.9) holds, a diffusion approximation is justified and equation (3.10) specifies the distribution of the approximating diffusion, $Z(\cdot)$. Using this expression, one can obtain the mean and variance of $Z(s)$ and thus, for large K, an approximate formula for the mean and variance of $X^{(K)}(s)$. If one denotes by ∇F the matrix of first partial derivatives of F, that is $\nabla F = [\partial F_i/\partial x_j]$, and puts $B_s = \nabla F(X(s,x_0))$, then

$$\mathbb{E}Z(s) = M_s z,$$

where

$$M_s = \exp\left(\int_0^s B_u du\right).$$

On the other hand, the covariance matrix, Σ_s, of $Z(s)$ is given by

$$\Sigma_s = M_s\left(\int_0^s M_u^{-1} G(X(u,x_0))(M_u^{-1})^T du\right) M_s^T.$$

It follows that

$$\operatorname{Cov} X^{(K)}(s) \simeq K^{-1}\Sigma_s$$

and a "working" approximation for the mean, obtained by setting z *equal to* $\sqrt{K}\left(X^{(K)}(0) - x_0\right)$, is given by

$$\mathbb{E}X^{(K)}(s) \simeq X(s, x_0) + M_s(X^{(K)}(0) - x_0).$$

Observe that the mean and variance of the numbers of circuits in use at time s are both of order K.

In the important special case where x_0 is chosen as an equilibrium point of the deterministic model, one can be far more precise in specifying the approximating diffusion. The equilibrium points of (4.1) have been studied extensively by Gibbens et al. (1990). They showed that if $x_0 = (p_0, p_1, \dots p_C)$ is an equilibrium point it must be of the form given by

$$p_j = \frac{\xi^j}{j!}\left(\sum_{i=0}^{C} \frac{\xi^i}{i!}\right)^{-1}, \qquad 0 \le j \le C,$$

where ξ solves

$$\xi = \nu + 2\nu E(\xi, C)\left(1 - E(\xi, C)\right). \tag{4.3}$$

The quantity $E(\xi, C)$, given by

$$E(\xi, C) = \frac{\xi^C}{C!}\left(\sum_{i=0}^{C} \frac{\xi^i}{i!}\right)^{-1},$$

is Erlang's formula for the loss probability of a single link with C circuits and with Poisson traffic offered at rate ξ. It is usually more convenient to calculate the solutions to (4.3) by setting $b = E(\xi, C)$ and solving the equation

$$b = E\left(\nu + 2\nu b(1-b), C\right); \tag{4.4}$$

this transformation of (4.3) shows that b could have been obtained as the celebrated Erlang Fixed Point of the model (see, for example, Kelly (1986)). For C sufficiently small, equation (4.4) has a unique solution and the corresponding equilibrium point is stable. However, if C is large enough, there can be two or even three solutions depending, then, on the magnitude of the ratio ν/C, and these give rise to two, respectively three, equilibrium points. In the case of two equilibrium points, one is stable and the other unstable, while in the case of three, two are stable and the other is unstable.

The following central limit law shows that the random fluctuations about any given equilibrium point, x_0, are Gaussian. Moreover, it shows that these fluctuations can be approximated by an Ornstein-Uhlenbeck (OU) process. It should be emphasised that x_0 need not be stable. Indeed, the approximation is appropriate for describing the fluctuations about the unstable equilibria.

Theorem 4.2. *Let x_0 be an equilibrium point of (4.1). Then, if*

$$\lim_{K\to\infty} \sqrt{K}\left(X^{(K)}(0) - x_0\right) = z,$$

the family of processes $\{Z^{(K)}(\cdot)\}$, defined by

$$Z^{(K)}(s) = \sqrt{K}\left(X^{(K)}(s) - x_0\right), \qquad 0 \le s \le t,$$

converges weakly in $D_E[0,t]$ to an Ornstein-Uhlenbeck process with local drift matrix $B = \nabla F(x_0)$, local covariance matrix $G = G(x_0)$, and with initial value $Z(0) = z$. In particular, $Z(s)$ is normally distributed with mean

$$\mu_s = e^{Bs} z$$

and covariance matrix

$$\Sigma_s = \int_0^s e^{Bu} G e^{B^T u} du = \Sigma - e^{Bs}\Sigma e^{B^T s},$$

where Σ, the stationary covariance matrix, satisfies

$$B\Sigma + \Sigma B^T + G = 0.$$

We can conclude that, for K large, $X^{(K)}(s)$ has an approximate normal distribution for each s, and an approximation for the mean and the covariance matrix of $X^{(K)}(s)$ is given by

$$\mathbb{E}X^{(K)}(s) \simeq x_0 + e^{Bs}(X^{(K)}(0) - x_0)$$

and

$$\operatorname{Cov} X^{(K)}(s) \simeq K^{-1}\left(\Sigma - e^{Bs}\Sigma e^{B^T s}\right).$$

The special case $C = 1$, where there is only one circuit available on each link, is exceedingly simple to analyse. Set

$$F(x) = 1 - (\nu+1)x - 2\nu(1-x)x^2, \qquad x \in (0,1),$$

and

$$G(x) = 1 + (\nu-1)x + 4\nu(1-x)x^2 \qquad x \in (0,1).$$

Then, it can be shown that F has a unique zero, x_0, on $(0,1)$, for all values of $\nu > 0$, this being a stable equilibrium point of the deterministic model, $dx/dt = F(x)$. If $X^{(K)}(s)$ is the proportion of links with no circuits in use at time s, then by virtue of the OU approximation, $X^{(K)}(s)$ has an approximate normal distribution with

$$\mathbb{E}X^{(K)}(s) \simeq x_0 + e^{Bs}(X^{(K)}(0) - x_0),$$

where $B = F'(x_0) = 6\nu x_0^2 - 4\nu x_0 - (\nu+1)(<0)$, and

$$\mathrm{Var}\, X^{(K)}(s) \simeq K^{-1}\frac{G(x_0)}{2B}(e^{2Bs}-1).$$

The magnitude of B, and hence the stability of x_0, increases as ν becomes large, but the stationary variance, $G(x_0)/(-2B)$, increases from 0 to a maximum around $\nu = 0.5$ and then decreases to 0.

In cases where $C > 1$, it is convenient (see Barbour (1976)) to employ a change of coordinates. If, as is the case envisaged here, the eigenvalues of $B = \nabla F(x_0)$ are real, an appropriate transformation is given by

$$W^{(K)}(s) = AZ^{(K)}(s),$$

where the rows of A are the left-eigenvectors of B. Since the column sums of B are all equal to 0, B has a zero eigenvalue, and so one of the components of $W^{(K)}$, say $W_0^{(K)}$, is identically zero, since because $\sum_{i=0}^{C} X_i^{(K)} = 1$, we have that $\sum_{i=0}^{C} Z_i^{(K)} = 0$. The sequence $\{W^{(K)}(\cdot)\}$, where for convenience $W^{(K)} = (W_1^{(K)}, W_2^{(K)}, \ldots W_C^{(K)})$, converges weakly to an OU process, $W(\cdot)$, whose individual components are, themselves, OU processes. Its local drift matrix is $D = \mathrm{diag}\,(\eta_1, \eta_2, \ldots \eta_C)$, where $\eta_1, \eta_2, \ldots \eta_C$ are the non-zero eigenvalues of B, and its local covariance matrix, S, is obtained from the matrix AGA^T by deleting the zeroth row and column. In particular, $W(s)$ has a properly C-dimensional normal distribution with

$$\mathbb{E}W_i(s) = w_i e^{\eta_i s},$$
$$\mathrm{Var}\, W_i(s) = \frac{S_{ii}}{2\eta_i}(e^{2\eta_i s}-1)$$

and

$$\mathrm{Cov}\,(W_i(s),\ W_j(s)) = \frac{S_{ij}}{\eta_i+\eta_j}(e^{(\eta_i+\eta_j)s}-1),$$

for $i = 1, 2\ldots, C$, where $w = Az$.

The change of coordinates allows us to use some powerful results of Barbour (1976) which establish asymptotic results on the time of first exit of $X^{(K)}(\cdot)$ from a region containing x_0. For example, suppose that x_0 is a stable equilibrium point and let $\tau(K, c_K)$ be the time when $W^{(K)}(\cdot)$ first crosses the contour

$$\left\{ w \in \mathbb{R}^C \ : \ \sum_{i=1}^{C} \sqrt{\frac{2T_{ii}}{w_i^2}} \exp\left(\frac{w_i^2}{2T_{ii}}\right) = c_K^{-1}\exp(c_K^2) \right\},$$

where T, the stationary covariance matrix of $W(\cdot)$, has elements $T_{ij} = -S_{ij}/(\eta_i+\eta_j)$ and $\{c_K\}$ is a sequence of real numbers such that $c_K \to \infty$ as $K \to \infty$; as Barbour notes, to order c_K^{-1}, the contour delimits the rectangle

$$\left\{ w \in \mathbb{R}^C \ : \ |w_i| \le c_K\sqrt(2T_{ii}),\ i = 1, 2, \ldots, C \right\}.$$

Then, Theorem 3 of Barbour (1976) states that if $c_K = o(K^{\frac{1}{8}})$, the random variable

$$-\tau(K, c_K)\frac{2}{\sqrt{\pi}}\eta c_K \exp(-c_K^2),$$

where $\eta = \sum_{i=1}^{C} \eta_i$, converges weakly to a unit-mean negative exponential random variable as $K \to \infty$. Thus, provided $c_K = o(K^{\frac{1}{8}})$, the time at which $W^{(K)}(\cdot)$ first crosses the contour is of order $c_K^{-1} \exp(c_K^2)$.

The result for the $C = 1$ case is more straightforward. Using Theorem 1(iii) of Barbour (1976), one can see that the time that $X^{(K)}(\cdot)$ first leaves the interval

$$\left\{x \ : \ |x - x_0| \leq K^{-\frac{1}{2}} c_K\right\}$$

is of order

$$\frac{1}{-2Bc_K}\sqrt{\frac{\pi G(x_0)}{-B}}e^{-Bc_K^2/G(x_0)},$$

whenever $c_K = o(K^{\frac{1}{8}})$. Hence, it is asymptotically larger than any power of K if, for example, $c_K = O(K^{\frac{1}{8}}/\log K)$.

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