ON THE RELATIONSHIP BETWEEN μ-INVARIANT MEASURES AND QUASISTATIONARY DISTRIBUTIONS FOR ABSORBING CONTINUOUS-TIME MARKOV CHAINS WHEN ABSORPTION IS NOT CERTAIN

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Abstract This note considers continuous-time Markov chains whose state space consists of an irreducible class, C, and an absorbing state which is accessible from C. The purpose is to provide results on μ-invariant and μ-subinvariant measures where absorption occurs with probability less than 1. In particular, the well known premise that the μ-invariant measure, m, for the transition rates be finite is replaced by the more natural premise that m be finite with respect to the absorption probabilities. The relationship between μ-invariant measures and quasistationary distributions is discussed.

1 INTRODUCTION

We consider the problem of obtaining quasistationary distributions for a continuous-time Markov chain directly from the q-matrix, Q, of transition rates. In Nair and Pollett (1993) it was shown that, for an arbitrary Q-process, a necessary and sufficient condition for a proper distribution, m, to be a quasistationary distribution is that m be a μ-invariant measure for the transition function. Thus, in view of Theorem 1 of Pollett (1986), which provides a necessary and sufficient condition for a μ-invariant measure for Q to be a μ-invariant measure for the *minimal* transition function, the problem of determining quasistationary distributions directly from the q-matrix is ostensibly solved, at least in the practically important case of the minimal process. However, the invariance condition, which is usually expressed in terms of the regularity or otherwise of a related q-matrix (see Theorem 3.1 of Pollett (1988)), is not easily verified in practice. Simpler conditions, which are much easier to check, are available under the premise that the μ-invariant measure, m, for Q be finite, that is, $\sum_{j\in C} m_j < \infty$; see Nair and Pollett (1993), Pollett and Vere-Jones (1992), van Doorn (1991) and Vere-Jones (1969). In view of the theory for the existence of limiting conditional distributions for evanescent Markov chains (see, for example, Flaspohler (1974)), a much more useful premise would be that m is finite with respect to the absorption probabilities, that is, m satisfies $\sum_{j\in C} m_j a_j < \infty$, where a_j is the probability that the process is eventually absorbed after starting in state j.

2 PRELIMINARIES

Let $S = \{0, 1, \ldots\}$ and let $Q = (q_{ij},\ i, j \in S)$ be a stable and conservative q-matrix over S. Suppose that P is a given Q-process, that is, a standard (but not necessarily honest) transition function, $(p_{ij}(\cdot),\ i, j \in S)$, with $p'_{ij}(0+) = q_{ij}$, over S. For C, a given subset of S, and μ some fixed non-negative real number, the measure $m = (m_j,\ j \in C)$ is said to be a *μ-subinvariant measure on C for P* if

$$\sum_{i\in C} m_i p_{ij}(t) \le e^{-\mu t} m_j, \qquad j \in C,\ t \ge 0,$$

and *μ-invariant* if equality holds for all $j \in C$ and $t \ge 0$. In contrast, m is said to be a *μ-subinvariant measure on C for Q* if

$$\sum_{i\in C} m_i q_{ij} \le -\mu m_j, \qquad j \in C,$$

and *μ-invariant* if equality holds for all $j \in C$.

Take $C = \{1, 2, \ldots\}$ and, for simplicity, suppose that C is irreducible for P, that is, for all $i, j \in C$, $p_{ij}(t) > 0$ for some (and then for all) $t > 0$. The irreducibility ensures that non-trivial μ-subinvariant measures, m, satisfy $m_j > 0$ *for all* $j \in C$. In addition, assume that 0 is an absorbing state, that is, $p_{0j}(t) = \delta_{0j}$, for $j \in S$ and $t > 0$; this is equivalent to assuming that $q_0 = 0$. Finally, assume that $q_{i0} > 0$ for at least one $i \in C$. This guarantees that there is a positive probability of absorption, that is, $p_{i0}(t) > 0$ for all $t > 0$.

3 INVARIANT VECTORS AND THE DUAL CHAIN

The key to our results is the notion of a dual chain and the application of the results of Nair and Pollett (1993) to that chain. The dual is usually defined in terms of an arbitrary invariant or subinvariant vector, but there will be no loss of generality in defining it in terms of the absorption probabilities, $a^P = (a_i^P,\ i \in S)$, given by $a_i^P = \lim_{t\to\infty} p_{i0}(t)$ for $i \in S$.

Let x be an invariant vector (on S) for P, that is, a collection of strictly positive numbers, $(x_j,\ j \in S)$, which satisfies

$$\sum_{j\in S} p_{ij}(t)x_j = x_i, \qquad i \in S,\ t \geq 0. \tag{3.1}$$

Suppose that $x_0 > 0$, so that under the above assumptions $x_j > 0$ for all $j \in C$. Define $\bar{P}(\cdot) = (\bar{p}_{ij}(\cdot),\ i,j \in S)$, the *dual of P with respect to x*, by

$$\bar{p}_{ij}(t) = p_{ij}(t)x_j/x_i, \qquad i,j \in S,\ t \geq 0. \tag{3.2}$$

Then, it is elementary to show that $\bar{P}$ is an honest transition function over S. Furthermore, if we divide (3.2) by t and let $t \downarrow 0$ we find that $\bar{p}'_{ij}(0+) = \bar{q}_{ij}$, where $\bar{q}_{ij} = q_{ij}x_j/x_i$, $i,j \in S$. Clearly, $\bar{q}_{ii} = q_{ii} > -\infty$, and, for $j \neq i$, $\bar{q}_{ij} \geq 0$. The q-matrix $\bar{Q} = (\bar{q}_{ij},\ i,j \in S)$ is called the *dual of Q with respect to x*. The following result demonstrates that $\bar{Q}$ is conservative.

Theorem 1 *x is invariant for Q, that is, $\sum_{j\in S} q_{ij}x_j = 0$ for $i \in S$.*

Proof. From (3.1) we have that

$$\sum_{j\neq i} \frac{p_{ij}(t)}{t} x_j = \frac{1 - p_{ii}(t)}{t} x_i, \qquad i \in S,\ t > 0,$$

and so, letting $t \downarrow 0$ and using Fatou's lemma, we deduce that x is *subinvariant* for Q, that is, $\sum_{j\in S} q_{ij}x_j \leq 0$ for $i \in S$. Hence $\sum_{j\in C} \bar{q}_{ij} \leq 0$ for $j \in C$, with equality if and only if x is invariant for Q. Now, since Q is conservative, P satisfies the backward equation

$$p'_{ij}(t) = \sum_{k\in S} q_{ik}p_{kj}(t), \qquad t \geq 0, \tag{3.3}$$

for all $i,j \in S$, from which it follows that $\bar{P}$ satisfies the corresponding backward equation

$$\bar{p}'_{ij}(t) = \sum_{k\in S} \bar{q}_{ik}\bar{p}_{kj}(t), \qquad t \geq 0, \tag{3.4}$$

for all $i, j \in S$. Thus, since $\bar{P}$ is honest we deduce, from Theorem II.17.2 of Chung (1967), that $\bar{Q}$ must be conservative and, hence, that x is invariant for Q. $\square$

In order to interpret conditions on $\bar{P}$ in terms of corresponding conditions on P, we shall need the following results, both of which follow directly from the definitions of μ-subinvariance and μ-invariance, and the definitions of $\bar{P}$ and $\bar{Q}$.

Lemma 2 *For any $i, j \in S$, P satisfies the forward equation*

$$p'_{ij}(t) = \sum_{k \in S} p_{ik}(t) q_{kj}, \qquad t \geq 0, \tag{3.5}$$

if and only if $\bar{P}$ satisfies the corresponding forward equation

$$\bar{p}'_{ij}(t) = \sum_{k \in S} \bar{p}_{ik}(t) \bar{q}_{kj}, \qquad t \geq 0. \tag{3.6}$$

Lemma 3 *Let m be a measure on C and define $\bar{m} = (\bar{m}_j,\ j \in C)$ by $\bar{m}_j = m_j x_j$. Then,*

(a) m is a μ-subinvariant measure on C for P if and only if $\bar{m}$ is a μ-subinvariant measure on C for $\bar{P}$, in which case m is μ-invariant if and only if $\bar{m}$ is μ-invariant.

(b) m is a μ-subinvariant measure on C for Q if and only if $\bar{m}$ is a μ-subinvariant measure on C for $\bar{Q}$, in which case m is μ-invariant if and only if $\bar{m}$ is μ-invariant.

We can now establish that there is no loss of generality in taking x to be a^P. Firstly $a_0^P = 1$. Next, since P satisfies the Chapman-Kolmogorov equations, we have, in particular, that

$$p_{i0}(t+s) = \sum_{k \in S} p_{ik}(t) p_{k0}(s), \qquad i \in C,\ s, t \geq 0, \tag{3.7}$$

and so $p_{i0}(t+s) \geq p_{i0}(t)$. Now, by assumption, $q_{j0} > 0$, for some $j \in C$, which, as already noted, implies that $p_{j0}(t) > 0$ for all $t > 0$. Hence $a_j^P > 0$. Letting $t \to \infty$ in (3.7) shows that a^P satisfies (3.1). Thus, in particular, $a_i^P \geq p_{ij}(t) a_j^P$ for all $i \in C$, from which it follows, by the irreducibility of C, that $a_i^P > 0$ for all $i \in C$. We have shown that a^P is a strictly positive invariant vector for P.

Theorem 4 *Let m be a μ-subinvariant measure on C for P which satisfies $\sum_{i \in C} m_i x_i < \infty$. Then, for all $i \in S$, $a_i^P = x_i / x_0$.*

Proof. As already noted, $\bar{P}$ is an honest $\bar{Q}$-process, and, from Lemma 3 (a), $\bar{m}$, given by $\bar{m}_j = m_j x_j$, $j \in C$, is a μ-subinvariant measure on C for $\bar{P}$. By the definition of $\bar{P}$, we have (in an obvious notation) that $a_i^{\bar{P}} = a_i^P x_0 / x_i$ for all $i \in S$. The result now follows by applying Lemma 3.1 of Nair and Pollett (1993) to $\bar{P}$: since $\bar{m}$ is finite and $\bar{P}$ is honest, $a_i^{\bar{P}} = 1$ for all $i \in C$. □

In the sequel, take $\bar{P}$ to be the dual of P with respect to a^P. It's q-matrix, $\bar{Q}$, will then be the dual of Q with respect to a^P. Note that C is irreducible for $\bar{P}$, that 0 is an absorbing state for $\bar{P}$, that is, $\bar{p}_{0j}(t) = \delta_{0j}$, and, that $a_i^{\bar{P}} = 1$.

4 RESULTS ON μ-INVARIANCE UNDER THE WEAKER PREMISE

In this section we shall show that all of the results contained in Sections 3 and 4 of Nair and Pollett (1993) hold good under the premise that $\sum_{j \in C} m_j a_j^P < \infty$. Observe that if, for some $\mu > 0$, there exists a *finite* μ-subinvariant measure on C for P, then, by Lemma 3.1 of Nair and Pollett (1993), $\sum_{j \in C} p_{ij}(t) \to 0$ as $t \to \infty$, for $i \in C$. Thus, if a finite μ-subinvariant measure exists, the Markov chain with transition function P will eventually leave C (with probability 1) if it starts there. So, for example, if P is honest and a_i^P is strictly less than 1 for some (and then for all) $i \in C$, then P admits no finite μ-subinvariant measures on C; all μ-subinvariant measures, m, must satisfy $\sum_{i \in C} m_i = \infty$.

Theorem 5 *Suppose that, for some $\mu > 0$, m is a μ-subinvariant measure on C for P which satisfies $\sum_{i \in C} m_i a_i^P < \infty$. Then,*

(a) if P satisfies the forward equations (3.5) for all $i \in C$ and $j = 0$,

$$\mu \sum_{i \in C} m_i a_i^P \leq \sum_{i \in C} m_i q_{i0}\,; \tag{4.1}$$

(b) if m is μ-invariant on C for P,

$$\mu \sum_{i \in C} m_i a_i^P \geq \sum_{i \in C} m_i q_{i0}. \tag{4.2}$$

Proof. Define $\bar{m} = (\bar{m}_j,\ j \in C)$ by $\bar{m}_j = m_j a_j^P$. Then, if P satisfies the forward equations (3.5) for all $i \in C$ and $j = 0$, $\bar{P}$ is a $\bar{Q}$-process which, by Lemma 2, satisfies the forward equations (3.6) for all $i \in C$ and $j = 0$. Since m is a μ-subinvariant measure on C for P, we have, by Lemma 3 (a), that $\bar{m}$ is a finite subinvariant measure on C for $\bar{P}$. Thus, we may apply Theorem 3.3 of Nair and Pollett (1993) to $\bar{P}$ and deduce that

$$\mu \sum_{i \in C} \bar{m}_i a_i^{\bar{P}} \leq \sum_{i \in C} \bar{m}_i \bar{q}_{i0}. \tag{4.3}$$

But, for all i, $a_i^{\bar{P}} = 1$ and $\bar{q}_{i0} = q_{i0}/a_i^P$, and so $\bar{m}_i \bar{q}_{i0} = m_i q_{i0}$. Thus, (4.3) is the same as (4.1).

If m is μ-invariant on C for P, then we have, by Lemma 3 (a), that $\bar{m}$ is a *finite* μ-invariant measure on C for $\bar{P}$. Thus, by applying Theorem 3.2 of Nair and Pollett (1993) to $\bar{P}$, we find that $\mu \sum_{i \in C} \bar{m}_i a_i^{\bar{P}} \geq \sum_{i \in C} \bar{m}_i \bar{q}_{i0}$, which is equivalent to (4.2). □

Corollary 6 *Suppose that P satisfies the forward equations (3.5) for all $i \in C$ and $j = 0$, and suppose that, for some $\mu > 0$, m is a μ-invariant measure on C for P which satisfies $\sum_{i \in C} m_i a_i^P < \infty$. Then,*

$$\mu \sum_{i \in C} m_i a_i^P = \sum_{i \in C} m_i q_{i0}. \tag{4.4}$$

The above results hold for any Q-process, in particular the *minimal* Q-process, that is the minimal solution, $F(\cdot) = (f_{ij}(\cdot),\ i, j \in S)$, to the backward equations. Thus, if m is a μ-invariant measure on C for F which satisfies $\sum_{i \in C} m_i a_i^F < \infty$, then (4.4) holds for F, because F satisfies the forward equations (3.5) for *every* $i, j \in S$. Our next result shows that this condition is also sufficient.

Theorem 7 *Let F be the minimal Q-process and suppose that, for some $\mu > 0$, m is a μ-subinvariant measure on C for Q which satisfies $\sum_{i \in C} m_i a_i^F < \infty$. Then, m is a μ-invariant measure on C for F if and only if*

$$\mu \sum_{i \in C} m_i a_i^F = \sum_{i \in C} m_i q_{i0}. \tag{4.5}$$

When this condition holds, m is μ-invariant on C for Q.

Proof. Define $\bar{m} = (\bar{m}_j,\ j \in C)$ by $\bar{m}_j = m_j a_j^F$. Then, since m is a μ-subinvariant measure on C for F, we have, by Lemma 3 (a), that $\bar{m}$ is a finite μ-subinvariant measure on C for $\bar{F}$, the dual of F. Thus, by Theorem 4.1 of Nair and Pollett (1993), a necessary and sufficient condition for $\bar{m}$ to be μ-invariant for $\bar{F}$, or equivalently, for m to be μ-invariant for F, is that

$$\mu \sum_{i \in C} \bar{m}_i a_i^{\bar{F}} = \sum_{i \in C} \bar{m}_i \bar{q}_{i0}. \tag{4.6}$$

But, $\bar{m}_i a_i^{\bar{F}} = \bar{m}_i = m_i a_i^F$ and $\bar{m}_i \bar{q}_{i0} = m_i q_{i0}$, and so (4.5) is the same as (4.6). Finally, if (4.5) holds (equivalently, (4.6) holds), Theorem 4.1 of Nair and Pollett (1993) also allows us to deduce that $\bar{m}$ is μ-invariant on C for $\bar{Q}$, from which it follows, by virtue of Lemma 3 (b), that m is μ-invariant on C for Q. □

5 QUASISTATIONARY DISTRIBUTIONS

The duality relationship identified in Section 3 leads to the following refinement of the notion of a quasistationary distribution. It is obtained by applying van Doorn's (1991) definition of a quasistationary distribution (see Definition 3.1 of Nair and Pollett (1993)) to the dual of P and then interpreting this for P itself. It reduces to van Doorn's definition in the case when $a_j^P = 1$ for all $j \in S$.

Definition 8 *Let $m = (m_j,\ j \in C)$ be a measure on C such that $\sum_{j \in C} m_j a_j^P = 1$, and define $h(\cdot) = (h_j(\cdot),\ j \in S)$ by*

$$h_j(t) = \sum_{i \in C} m_i p_{ij}(t), \qquad j \in S,\ t > 0,$$

and $\rho = (\rho_j,\ j \in C)$ by

$$\rho_j = m_j a_j^P, \qquad j \in C. \tag{5.1}$$

Then, ρ is a quasistationary distribution on C for P if, for all $t > 0$ and $j \in C$,

$$\frac{h_j(t) a_j^P}{\sum_{i \in C} h_i(t) a_i^P} = \rho_j. \tag{5.2}$$

Thus, if $(X(t),\ t \geq 0)$ is a continuous-time Markov chain with transition function P, then, when $h(t)$ can be interpreted as the absolute distribution of the process at time t, ρ is a quasistationary distribution if and only if

$$\Pr\left[X(t) = j | X(t) \in C, X(t+s) = 0,\ \text{for some } s > 0\right] = \rho_j, \qquad j \in C,$$

that is, the state probabilities at time t, conditional on both the process being in C at t *and* the process eventually reaching 0, are the same for all t. If $\sum_{i \in C} m_i = 1$, then m can be interpreted as the initial distribution of the process, but note that $\sum_{i \in C} m_i < \infty$ is not necessary for $h(t)$ to be interpreted as its absolute distribution at time t; see Sections 2 and 4 of Reuter (1962), as well as Lamb (1971).

We can identify a relationship between this general notion of a quasistationary distribution and μ-invariant measures for P. Let $\bar{P}$ be the dual of P and define $\bar{p}(\cdot) = (\bar{p}_j(\cdot),\ j \in S)$ by $\bar{p}_j(t) = h_j(t)a_j^P$, $j \in S$, $t > 0$, so that $\bar{p}_j(t) = \sum_{i\in C} \rho_i \bar{p}_{ij}(t)$, $j \in S$, $t > 0$, that is, $\bar{p}(t)$ is the absolute distribution at time t of a continuous-time Markov chain with transition function $\bar{P}$ and initial distribution ρ. It follows that (5.2) can be written as $\bar{p}_j(t)/\sum_{i\in C} \bar{p}_i(t) = \rho_j$. Thus, if $m = (m_j,\ j \in C)$ is a measure on C such that $\sum_{j\in C} m_j a_j^P = 1$, then ρ, given by (5.1), is a proper distribution over C, and, since $a_j^{\bar{P}} = 1$ for all $j \in S$, ρ is a quasistationary distribution on C for P if and only if it is a quasistationary distribution on C for $\bar{P}$. By Proposition 3.1 of Nair and Pollett (1993), this holds if and only if, for some $\mu > 0$, ρ is μ-invariant on C for $\bar{P}$, or equivalently, by Lemma 3 (a), m is μ-invariant on C for P. Thus, we have proved the following result:

Theorem 9 *Let $m = (m_j,\ j \in C)$ be a measure on C such that $\sum_{j\in C} m_j a_j^P = 1$ and define $\rho = (\rho_j,\ j \in C)$ by (5.1). Then, ρ is a quasistationary distribution on C for P if and only if, for some $\mu > 0$, m is μ-invariant on C for P.*

Using Theorem 9, together with Theorem 3.1 of Nair and Pollett (1993), we arrive at the following corollary, which identifies a relationship between μ-invariant measures on C for Q and the more general notion of a quasistationary distribution for P:

Corollary 10 *Let $m = (m_j,\ j \in C)$ be a measure on C such that $\sum_{j\in C} m_j a_j^P = 1$ and suppose that $\rho = (\rho_j,\ j \in C)$, defined by (5.1), is a quasistationary distribution on C for P. Then, for some $\mu > 0$, m is a μ-subinvariant measure on C for Q and μ-invariant if and only if P satisfies the forward equations (3.5) for all $i, j \in C$.*

Example We shall illustrate our results with reference to the absorbing birth-death process on $S = \{0, 1, \ldots\}$. This has transition rates given by

$$q_{ij} = \begin{cases} \lambda_i, & \text{if } j = i+1, \\ -(\lambda_i + \mu_i), & \text{if } j = i, \\ \mu_i, & \text{if } j = i-1, \\ 0, & \text{otherwise,} \end{cases}$$

where the birth rates $(\lambda_i,\ i \geq 0)$ and the death rates $(\mu_i,\ i \geq 0)$ satisfy $\lambda_i,\ \mu_i > 0$, for $i \geq 1$, and $\lambda_0 = \mu_0 = 0$. Thus, 0 is an absorbing state and $C = \{1, 2, \ldots\}$ is an irreducible class. Define $\pi = (\pi_i,\ i \geq 1)$ by $\pi_1 = 1$, and $\pi_i = \prod_{j=2}^{i}(\lambda_{j-1}/\mu_j)$ for $i \geq 2$. Noting that $\lambda_n \pi_n = \mu_{n+1}\pi_{n+1}$ for all $n \geq 0$, then define

$$\boldsymbol{A} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n \pi_n} \quad \text{and} \quad \boldsymbol{C} = \sum_{m=1}^{\infty} \pi_m \sum_{n=m}^{\infty} \frac{1}{\lambda_n \pi_n}$$

Note also that Q is regular if and only if $\boldsymbol{C} = \infty$.

The spectral representation approach to birth and death processes, introduced by Karlin and McGregor (1957a), involves constructing a sequence of orthogonal polynomials, $\mathcal{Q}(\cdot) = (\mathcal{Q}_i(\cdot),\ i \in C)$, with $\mathcal{Q}(s)$ being an s-invariant vector on C for Q, that is, defined by $\mathcal{Q}_1(s) = 1$, $-s\mathcal{Q}_1(s) = \lambda_1 \mathcal{Q}_2(s) - (\lambda_1 + \mu_1)\,\mathcal{Q}_1(s)$ and $-s\mathcal{Q}_i(s) = \lambda_i \mathcal{Q}_{i+1}(s) - (\lambda_i + \mu_i)\,\mathcal{Q}_i(s) + \mu_i \mathcal{Q}_{i-1}(s)$ for $i \geq 2$. Any notational occurrence of $\mathcal{Q}_0(\cdot)$ should be interpreted as zero, and for all $i \geq 2$, $\mathcal{Q}_i(0) = 1 + \mu_1 \sum_{j=1}^{i-1} 1/(\lambda_j \pi_j)$, whilst $\mathcal{Q}_\infty(0) := \lim_{i\to\infty} \mathcal{Q}_i(0) < \infty$ if and only if $\boldsymbol{A} < \infty$.

The probabilities of absorption of the minimal process, into state 0, given the process starts from some state $i \in S$, that is, $a^F = (a_i^F,\ i \in S)$, are well known (see Karlin and McGregor (1957b)): $a_i^F = 1$ for all i if and only if $\boldsymbol{A} = \infty$, and, when $\boldsymbol{A} < \infty$, $a_i^F = 1 - \mathcal{Q}_i(0)/\mathcal{Q}_\infty(0)$ for $i \in S$.

It is also well known that π is a subinvariant measure on C for Q, and hence, from Theorem 4.1 b(ii) of Pollett (1988), that, for any $\mu \in [0, \lambda]$, where λ is the decay parameter of C, $m = (\pi_i \mathcal{Q}_i(\mu),\ i \in C)$ is the unique μ-invariant measure on C for Q.

Assuming that $\boldsymbol{C} = \infty$, that is Q is regular, and that $\boldsymbol{A} < \infty$, that is $a_i^F < 1$, it then follows, from Theorem 3.5 (ii) of Kijima et al. (1997), that

$$\sum_{i=1}^{\infty} m_i a_i^F = \sum_{i=1}^{\infty} \pi_i \mathcal{Q}_i(\mu) \left\{ 1 - \frac{\mathcal{Q}_i(0)}{\mathcal{Q}_\infty(0)} \right\} < \infty.$$

Moreover, the only invariant vector, x, on S for F such that $\sum_{i\in C} m_i x_i < \infty$ is a^F, as stated in Theorem 4. It also follows, from Theorem 7, that m is a μ-invariant measure on C for F if and only if

$$\frac{\mu}{\mu_1} \sum_{i=1}^{\infty} \pi_i \mathcal{Q}_i(\mu) \left\{ 1 - \frac{\mathcal{Q}_i(0)}{\mathcal{Q}_\infty(0)} \right\} = 1.$$

Thus, by Theorem 9, $\rho = (\rho_i,\ i \in C)$, given by $\rho_i = \frac{\mu}{\mu_1}\pi_i Q_i(\mu)\{1 - Q_i(0)/Q_\infty(0)\}$, is a quasistationary distribution on C for F. This result is now well known; for a detailed analysis and extensions to the case when Q is not regular, see Kijima et al. (1997).

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