Bounds for the Decay Parameter of a Birth-Death Process

Hanjun Zhang, David Sirl and Phil Pollett

THE UNIVERSITY OF QUEENSLAND
AUSTRALIA
Department of Mathematics

AUSTRALIAN RESEARCH COUNCIL
Centre of Excellence for Mathematics and Statistics of Complex Systems

Absorbing Markov Processes

The problem we will consider relates to Markov processes in continuous time on a countable state space S. We recall that the fundamental analytical tool for such processes is the transition function $P(t) = (p_{ij}(t),\ i,j \in S, t > 0)$, where $p_{ij}(t)$ represents the probability that the process, having started in state i, is in state j after elapsed time t. However we rarely know the transition function explicitly and must extract what information we can out of the so-called *q-matrix* of transition rates: $Q = (q_{ij} = p'_{ij}(0^+),\ i,j \in S)$. Here we will assume that we are given a stable conservative q-matrix Q satisfying $q_{0j} = 0$ for all $j \in S$ (so that state zero is absorbing) and for which the remaining states $C = S \setminus \{0\}$ comprise an irreducible transient class. In addition we assume that the absorbing state is reached with probability one. We will take C to be either $\{1, 2, \ldots, N\}$ or $\{1, 2, \ldots\}$ according as we require S to be finite or countably infinite.

Birth-Death Processes

We also recall that a Markov process on a subset of the integers is called a birth-death process if $q_{ij} = 0$ whenever $|i - j| > 1$. As usual we denote the 'birth rates' $q_{i,i+1}$ by λ_i, the 'death rates' $q_{i,i-1}$ by μ_i. To ensure that we have the desired class structure in the state space, we must have $\lambda_0 = 0$ and both λ_i and μ_i positive for all $i \geq 1$. The simple structure of birth-death processes means that they submit to analysis rather more easily than more general processes. In addition to this, a great many of the Markov chain models used in practice are in fact birth-death processes.

The Decay Parameter

The decay parameter is a very important quantity in the study of absorbing Markov processes. It can be defined by the limit

$$\lambda_C = \lim_{t \to \infty} -t^{-1} \log(p_{ij}(t)),$$

which exists and is the same for all $i, j \in C$ [2]. The decay parameter is of central importance in the theory of quasi-stationary and limiting conditional distributions (in fact the precise value is not so important here, only whether it is positive or zero), but we will be particularly interested in it because if an absorbing Markov process has its quasi-stationary distribution as its initial distribution, then the time to absorption, is exponentially distributed with parameter λ_C.

This makes the decay parameter a quantity of great interest to those who use absorbing Markov processes in modelling, for example epidemiologists and population ecologists. Unfortunately the decay parameter is notoriously difficult to evaluate or even approximate and this has led to considerable effort being devoted to the task of approximating the decay parameter for several particular models which are useful in applications, usually in the guise of the expected time to extinction starting in the quasi-stationary distribution — the recipocal of λ_C.

The Decay Parameter of an Absorbing Birth-Death Process

Although it is in general not possible to evaluate the decay parameter of an absorbing Markov process, there are some partial results for birth-death processes. van Doorn [4] gives some variational formulae for the decay parameter from which he obtains upper and lower bounds, but getting good bounds requires a good approximation of the quasi-stationary distribution, which is rarely available. A major advance is provided by Chen [1], who gives upper and lower bounds for the decay parameter of a general birth-death process which always differ by a factor of four.

Theorem (Chen [1]): Suppose that a birth-death process on the state space $\{0\} \cup \{1, 2, \ldots, N\}$ has birth and death rates $(\lambda_i,\ 1 \leq i \leq N-1)$ and $(\mu_i,\ 1 \leq i \leq N)$, and put $\pi_1 = 1$ and $\pi_i = \prod_{j=2}^{i} \lambda_{j-1}/\mu_j,\ 2 \leq i \leq N$. Now define

$$R_n = \left(\frac{1}{\mu_1} + \sum_{i=1}^{n-1} \frac{1}{\lambda_i \pi_i} \right) \sum_{i=n}^{N} \pi_i$$

and $S = \max_{1 \leq n \leq N} R_n$. Then $(4S)^{-1} \leq \lambda_C \leq S^{-1}$.

We note that Chen's result is in fact for birth-death process on the infinite state space $\{1, 2, \ldots\}$, where but for 'N' being replaced by '∞' and '$\max$' by '$\sup$', the formulae remain the same. From this immediately follows the corollary that $\lambda_C > 0$ if and only if $\sup R_n < \infty$, both of which are trivially true when $N < \infty$.

We will compare the bounds given by the above theorem with approximations derived by Nåsell for the well-known *stochastic logistic* or *SIS epidemic* model, which is a birth-death process on $S = \{0\} \cup \{1, 2, \ldots, N\}$ with birth and death rates

$$\lambda_i = \frac{\lambda i}{N}(N - i) \quad \text{and} \quad \mu_i = \mu i.$$

Nåsell has refined his methods in a series of papers, the most recent being [3] which contains a good survey of the problem of approximating the quasi-stationary distribution and the expected time to extinction both from quasi-stationarity and from a fixed state, for the stochastic logistic model.

Numerical Results

We observe and compare the behaviour of the approximations of the decay parameter obtained using Matlab implementations of both Nåsell's methods, the bounds of Theorem 1 relative to the true value calculated using Matlab's `eigs` routine. In line with Nåsell we focus our attention on the more interesting parameter values $R = \lambda/\mu > 1$, when the process usually remains extant for long periods of time before being absorbed, and also observe that the behaviour of the approximations and bounds appears to depend on λ and μ only through R.

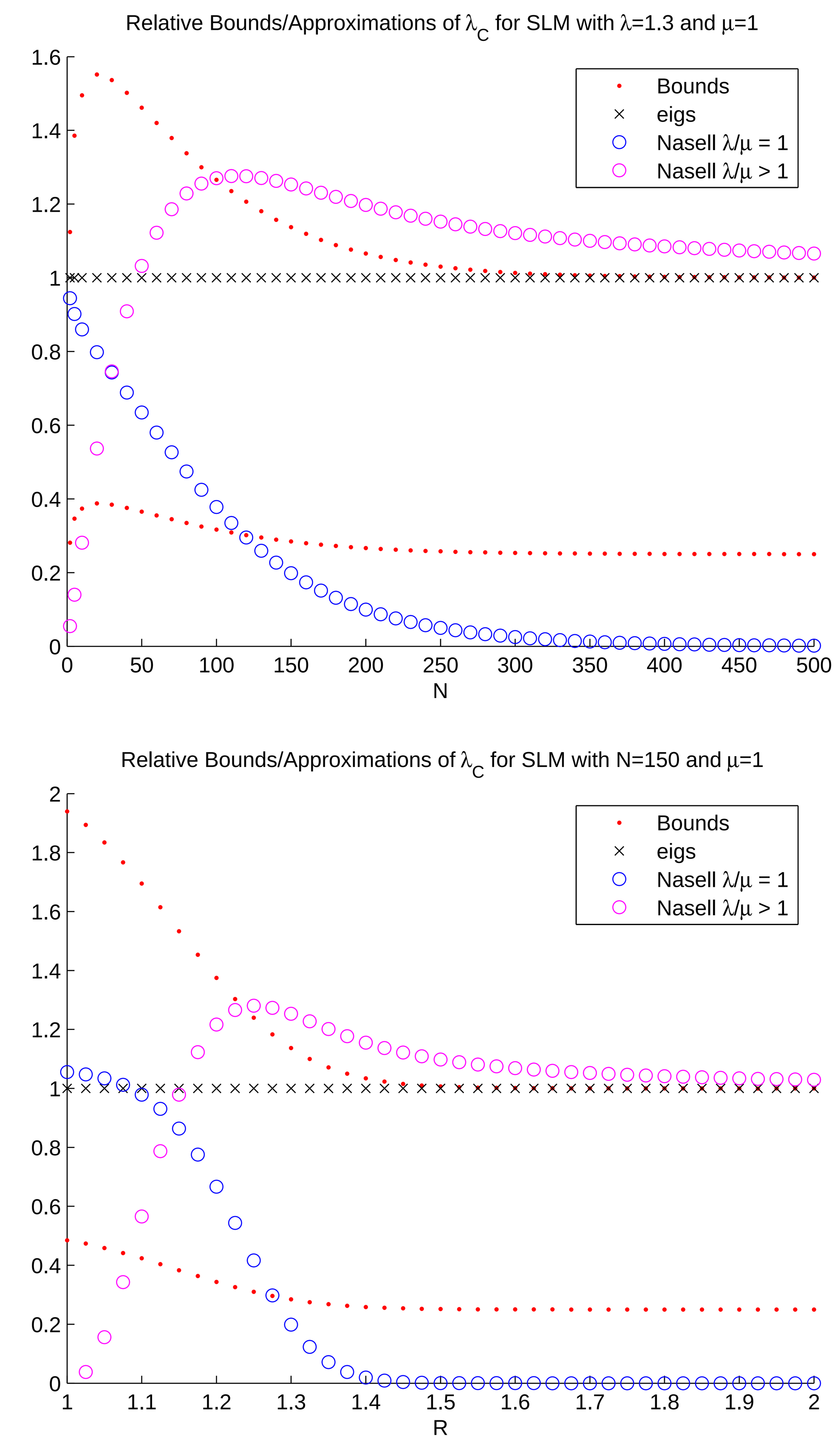

Figure 1: Relative bounds and approximations both for fixed R and varying N, and for fixed N and varying R.

Further Numerical Results and Discussion

We can see from Figure 1 that both Nåsell's approximations and the upper bound of Theorem 1 become better approximations for the decay parameter as both N and R become larger. We have also observed that the relative behaviour of the approximations and bounds appears to depend on λ and μ almost exclusively through their ratio R.

In particular, on both plots in Figure 1 it seems that the upper bound is a better approximation to the decay parameter than Nåsell's approximation. In addition, Figure 2 shows a large region in (N, R)-space where the relative error of the upper bound (as an approximation of λ_C) is less than 10^{-3}.

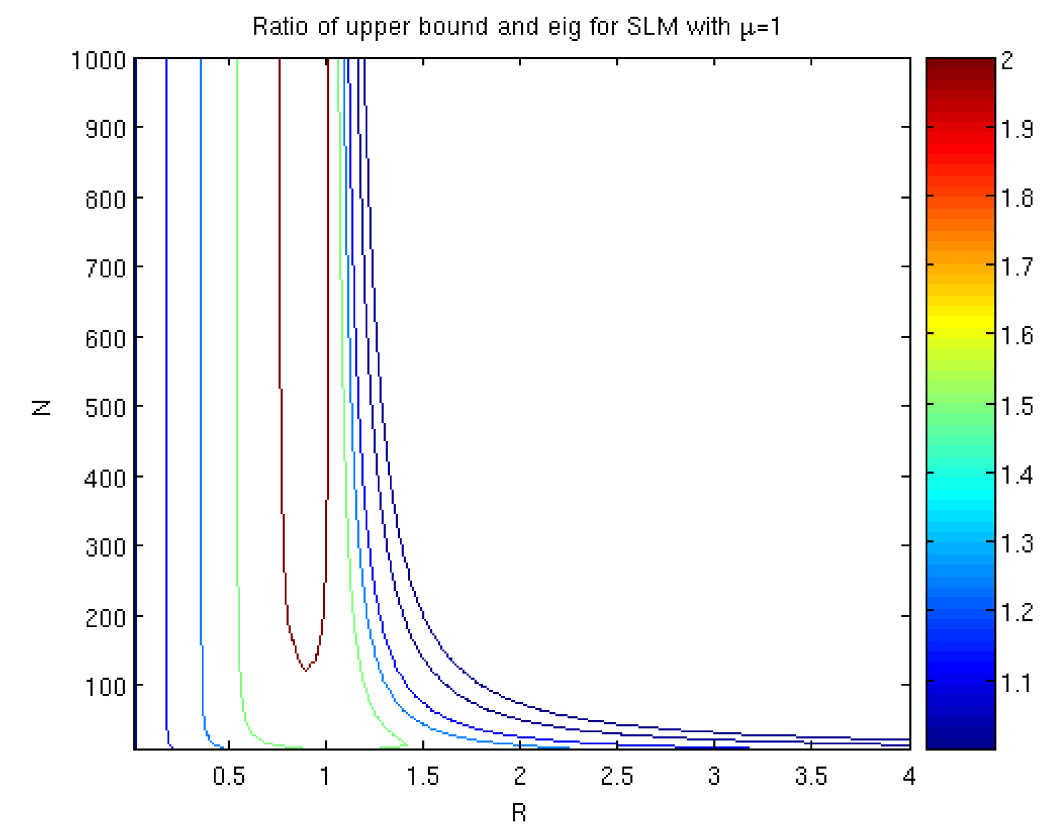

Figure 2: A contour plot of the ratio S/λ_C.
The contours are for the values 2, 1.5, 1.25, 1.1, 1.01 and 1.001.

Using the upper bound as an approximation also offers the distinct advantage of being sure that we are overestimating λ_C, which from the applied perspective means underestimating the time to extinction from quasi-stationarity. Especially in an ecological setting, this is far preferable to overestimating the extinction time, as an underestimate results in a more conservative assessment of a populations viability.

The bounds given in Theorem 1 are thus likely to be of significant interest to modellers using absorbing birth-death processes as they give accurate approximations of the decay parameter, with errors known to be negative.

References

[1] Chen, Mu-Fa, Explicit bounds of the first eigenvalue, *Sci. China Ser. A*, 43(10):1051–1059, 2000.

[2] Kingman, J. F. C., The exponential decay of Markov transition probabilities, *Proc. London Math. Soc.*, 13:337–358, 1963.

[3] Nåsell, Ingemar, Extinction and quasi-stationarity in the Verhulst logistic model, *J. Theor. Biol.*, 211:11–27, 2001. Extended version at `www.math.kth.se/ingemar/forsk/verhulst/verhulst.html`

[4] van Doorn, E. A., Conditions for exponential ergodicity and bounds for the decay parameter of a birth-death process, *Adv. Appl. Prob.*, 17(3):514–530, 1985.