Quasi-stationary distributions and the decay parameter

Hanjun Zhang

Department of Mathematics, University of Queensland

hjz@maths.uq.edu.au

AUSTRALIAN RESEARCH COUNCIL
Centre of Excellence for Mathematics
and Statistics of Complex Systems

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Introduction

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We denote by $P = (p_{ij}(t))$ transition probabilities of the minimal process which here is the unique process with generator Q.
Let $\mathbf{P_i}(\cdot) = \mathbf{P_i}(\cdot \mid \mathbf{X_0} = \mathbf{i})$ and If ν is a finite measure on $\mathbb{N}$, let $\mathbf{P}_\nu = \sum \nu_\mathbf{i} \mathbf{P_i}$. Here and below any unqualified sum is taken over $\mathbb{N}$.

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Finally, assume that $\mathbb{N}$ is irreducible and that 0 is accessible from some (and hence from every) state in $\mathbb{N}$.

Introduction

We further define

$$T = \inf\{t \geq 0 : X(t) = 0\}$$

the absorption (hitting) time at 0. We shall only the interested in processes for which $\mathbf{E_i T} < \infty$ for all $i \geq 1$.

Quasi-stationary distributions

A **quasi-stationary distribution (qsd)** $M = (m_i)$ is a probability measure on $\{1, 2, \cdots\}$ with the property that, starting with $M = (m_i)$, the conditional distribution, given the event that at time t the process has not been absorbed, still $M = (m_i)$. That is,

$$\frac{\sum m_i \mathbf{P_i}(\mathbf{X}(\mathbf{t}) = \mathbf{j})}{\sum m_i \mathbf{P_i}(\mathbf{X}(\mathbf{t}) \neq \mathbf{0})} = m_j. \quad (1)$$

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Quasi-stationary distributions for Markov processes and chains have been studied by several authors. Vere-Jones (1962), Seneta and Vere-Jones (1996) and Kingman (1963) studied the case of a general denumerable state space.

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Actually, there are a great deal of papers (almost over 300 papers) dealing with the **qsd**s

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Given $\mu \geq 0$ we call a measure M on $\mathbb{N}$ a **μ-invariant measure for Q** if for each $j \geq 1$,

$$\sum m_i q_{ij} = -\mu m_j, \quad (6)$$

and if for all $t > 0$,

$$\sum m_i p_{ij}(t) = e^{-\mu t} m_j, \quad (7)$$

it is called **μ-invariant on $\{1, 2, \cdots\}$ for P**.

Quasi-stationary distributions

M.G. Nair and P.K.Pollett (1993)show that if M is probability distribution on $\{1, 2, \cdots\}$. Then M is a quasi-stationary distribution on $\{1, 2, \cdots\}$ for P if and only if, for some $\mu > 0$, M is μ-invariant on $\{1, 2, \cdots\}$ for P.

Quasi-stationary distributions

We call $M = (m_j)$ **ν- the limit conditional distribution (ν-LCD)** if ν is a probability measure on $\{1, 2, \cdots\}$ and each $j \geq 1$

$$m_j = \lim_{t \to \infty} \mathbf{P}_\nu(\mathbf{X_t} = \mathbf{j} \mid \mathbf{T} > \mathbf{t}) \tag{8}$$

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The ν-LCD is a **qsd** (Vere-Jones(1996)).

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(i) determination of all **qsd**'s; and

(ii) solve the domian of attraction problem, namely, characterize all probability measure ν such that a given **qsd** M is a ν-LCD.

Although (i) has been addressed for several cases, details about (ii) are known only for finite Markov processes, and for the subcritical MBP.

Quasi-stationary distributions

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P.A. Ferrari, H. Kesten, S. Martinez and P. Picco (1995) prove the following interesting result which makes no reference to this general theory. They make the following definition of *asymptotic remoteness* (AR) of the absorbing state: For each $t > 0$

$$\lim_{i \to \infty} \mathbf{P_i}(\mathbf{T} > \mathbf{t}) = \mathbf{1}. \tag{12}$$

In other words, $T \Rightarrow \infty$ as $i \to \infty$.

Quasi-stationary distributions

Assume that AR condition holds, Ferrari et al. prove that a **qsd** exists iff

$$\mathbf{E_i}(\mathbf{e}^{\epsilon \mathbf{T}}) < \infty \tag{13}$$

for some $\epsilon > 0$ and $i \in \mathbb{N}$.

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Assume that AR condition holds, Ferrari et al. prove that a **qsd** exists iff

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Indeed this condition is necessary with, or without, AR condition.

Quasi-stationary distributions

Assume that AR condition holds, Ferrari et al. prove that a **qsd** exists iff

$$\mathbf{E_i}(\mathbf{e}^{\epsilon \mathbf{T}}) < \infty \tag{15}$$

for some $\epsilon > 0$ and $i \in \mathbb{N}$.

Indeed this condition is necessary with, or without, AR condition.

T. G. Pakes (1994) investigates what happens in a number of examples when AR condition fails. In fact, he examine quite closely two examples which violate AR condition but which nevertheless can have a **qsd** , showing AR condition is far from being a necessary condition, though it seems essential for the proofs of Ferrari et al.'s theorem.

Quasi-stationary distributions

First we have the following

Proposition 1 The following statements are equivalent:

1. Equation (22) holds, that is,

   $$\mathbf{E_i}(\mathbf{e}^{\epsilon \mathbf{T}}) < \infty$$

   for some $\epsilon > 0$ and $i \in \mathbb{N}$.

2. There is λ with $0 < \lambda < \inf_{i \geq 1} q_i$ (here $q_i \equiv -q_{ii}$) for which the system

   $$\sum_{j \neq i} q_{ij} x_j \leq (q_i - \lambda) x_i - 1, \quad i \geq 1, \quad x_0 = 0 \qquad (16)$$

   has a finite non-negative solution.

Quasi-stationary distributions

Secondly we can obtain that the following condition

$$\lim_{i \to \infty} \mathbf{E_i T} = \infty \tag{17}$$

can substitute for the AR condtion (that is, for each $t > 0$ $\lim_{i \to \infty} \mathbf{P_i}(\mathbf{T} > \mathbf{t}) = \mathbf{1}$.) which preserves the main result of Ferrari et all (1995).

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2. As we know, the mean extinction time $\mathbf{E_i T}$ is the minimal non-negative solution of the system

$$\sum_{j \geq 0} q_{ij} z_j = -1, \quad i \geq 1, \quad z_0 = 0. \tag{19}$$

So condition (17) is easier to check than AR condition.

Quasi-stationary distributions

Our main result is as follows

Theorem 1 Assume that Q is stable, conservative and regular, and that Q restricted to $\{1, 2, \cdots\}$ is irreducible. Assume further that (17) holds, that is

$$\lim_{i \to \infty} \mathbf{E_i T} = \infty$$

and that $\mathbf{P_i(T < \infty) = 1}$ for some (and hence all) i. Then a necessary and sufficient condition for the existence of a **qsd** is that there is λ with $0 < \lambda < \inf_{i \geq 1} q_i$ for which the system (16)(that is, $\sum_{j \neq i} q_{ij} x_j \leq (q_i - \lambda) x_i - 1, \ \ i \geq 1, \ \ x_0 = 0$) has a finite non-negative solution.

The decay parameter

Suppose we have a q-matrix Q over E. Let P be an arbitrary Q-transition function. Suppose that $E = \{0\} \cup C$, where 0 is an absorbing state and $C = \{1, 2, \cdots\}$ is irreducible. The decay parameter λ is defined by

$$\lambda = \lim_{t \to \infty} -\frac{1}{t} \log P_{ij}(t).$$

Kingman showed that this limit exists and is the same for all $i, j \in C$, and that $0 \leq \lambda < \infty$.

The decay parameter

It is called the **decay parameter** because there exist constants $M_{ij} > 0$ with $M_{ii} = 1$ such that

$$P_{ij}(t) \le M_{ij} e^{-\lambda t}, \qquad i, j \in C.$$

Note, in particular, that $P_{ii}(t) \le e^{-\lambda t}$.

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$\lambda = \sup \{\alpha : P_{ij}(t) = O(\exp[-\alpha t]) \; as \; t \to \infty \; \forall i, j \in C\}$.

If $\mu > \lambda$, there does not exist any μ-invariant measure (Pollett 1986); in particularly, there does not exist any **qsd** if $\lambda = 0$.

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- to give criteria for decay parameter λ to be positive in terms of the rates (q_{ij});
- to determine the value of λ, or at least bounds for λ, in terms of the rates (q_{ij}).

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Example 1 Markov branching process (MBP).

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We shall adopt the usual notation (Anderson (1991))in prescribing MBP, that is, let $p_k, k \geq 0$, denote a sequence of non-negative numbers such that $\sum_{k=0}^{\infty} p_k = 1$, and let

$$p(s) = \sum_{k=0}^{\infty} p_k s^k, \quad 0 \leq s \leq 1,$$

denote the probability generating function of this sequence.

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$m = p'(1) = \sum_{k=0}^{\infty} kp_k$.

The decay parameter

MBP with generator Q is given by

$$q_{ij} = \begin{cases} 0 & \text{if } \; j < i - 1 \\ -ia(1 - p_i) & \text{if } \; j = i \\ iap_{j-i+1} & \text{if } \; j \geq i - 1, j \neq i. \end{cases}$$

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It is well known that $\mathbf{P_i}(\mathbf{T} < \infty) = \mathbf{1}$ iff $m \leq 1$.

And if $m \leq 1$, then the decay parameter is $\lambda = (1-m)a$.

The decay parameter

Example 2: The birth and death process

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We shall adopt the usual notation in prescribing birth rates $\lambda_i > 0 \; (i \geq 1)$, with $\lambda_0 = 0$, and death rates $\mu_i > 0 \; (i \geq 1)$. Now define by $\pi_1 = 1$ and

$$\pi_n = \prod_{k=2}^{n} \frac{\lambda_{k-1}}{\mu_k}, \qquad n \geq 2.$$

We will assume the process is absorbed with probability 1, that is,

$$\sum_{n=1}^{\infty} \frac{1}{\pi_n \lambda_n} = \infty. \tag{21}$$

The decay parameter

In order to state our main results, we need the following notation:

$$Q_n = \left(\frac{1}{\pi_1 \mu_1} + \sum_{j=1}^{n-1} \frac{1}{\pi_j \lambda_j} \right) \sum_{j=n}^{\infty} \pi_j, \quad n \geq 1,$$

and

$$S_0 = \sup_{n \geq 1} \; Q_n.$$

The decay parameter

Phil Pollett and Hanjun Zhang have obtained the following

Theorem 2 $(4S_0)^{-1} \leq \lambda \leq S_0^{-1}$.

And, hence,

$$\lambda > 0 \quad if\ and\ only\ if \quad S_0 < \infty.$$

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Proposition 2 If

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for some $\epsilon > 0$ and $i \in \mathbb{N}$, then the decay parameter $\lambda > 0$.

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For a general Markov chain, we have the following

Proposition 2 If

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By using **Propsosition 1**, we get

The decay parameter

Theorem 3 If there is λ with $0 < \lambda < \inf_{i \geq 1} q_i$ for which the system

$$\sum_{j \neq i} q_{ij} x_j \leq (q_i - \lambda) x_i - 1, \quad i \geq 1, \quad x_0 = 0 \tag{25}$$

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We have the following interesting result:

Corollary If $\sup_{i \geq 1} \mathbf{E_i T} < \infty$, then $\lambda > 0$.

The GMBP

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where, as above, $p_k, k \geq 0$, denote a sequence of non-negative numbers such that $\sum_{k=0}^{\infty} p_k = 1$.

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It can be seen that the ordinary MBP corresponds to the special case of $\nu = 1$.

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(iii) Assume the given GMBP Q is regular. Then the extinction probability of the corresponding GMBP is 1 if and only if $m \leq 1$.

Recall that a conservative Q is called regular if the Feller minimal Q-process is honest and thus there exists unique Q-process.

The GMBP

Anyue Chen (2002) obtained the following conclusion

If assume that the probability of eventual extinction is 1, i.e., assume $m \leq 1$. Then for all $i \geq 1$, $\mathbf{E_i T}$ are finite if and only if

$$\int_0^1 \frac{1-y}{p(s)-s}(-\ln y)^{\nu-1} dy < \infty. \qquad (28)$$

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Moreover, If (30) is true, then for all $i \geq 1$

$$E_i T = \frac{1}{\Gamma(\nu)} \int_0^1 \frac{1-y}{a(p(s)-s)}(-\ln y)^{\nu-1} dy < \infty. \tag{31}$$

Where $\Gamma(\nu)$ is the gamma function.

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(ii) If $m = 1$ and $\nu \geq 2$, then the decay parameter $\lambda > 0$.

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Theorem 3 (i) If $m < 1$, and $\nu \geq 1$, then the decay parameter $\lambda > 0$.

(ii) If $m = 1$ and $\nu \geq 2$, then the decay parameter $\lambda > 0$.

(iii) If $m = 1$, $1 < \nu \leq 2$ and $\sum_{k=1}^{\infty} k^2 p_k < \infty$, then there exists a **qsd**.

Conclusion

Theorem 1 Assume that Q is stable, conservative and regular, and that Q restricted to $\{1, 2, \cdots\}$ is irreducible. Assume further that (17) holds, that is

$$\lim_{i\to\infty} E_i T = \infty$$

and that $\mathbf{P_i(T < \infty) = 1}$ for some (and hence all) i. Then a necessary and sufficient condition for the existence of a **qsd** is that there is λ with $0 < \lambda < \inf_{i\geq 1} q_i$ for which the system (16), that is,

$$\sum_{j\neq i} q_{ij} x_j \leq (q_i - \lambda) x_i - 1, \quad i \geq 1, \quad x_0 = 0$$

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The decay parameter plays an important role in studying properties of Markov chains.

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has a finite non-negative solution, then the decay parameter $\lambda > 0$.

Corollary If $\sup_{i\geq 1} \mathbf{E_i T} < \infty$, then $\lambda > 0$.

Conclusion

For every birth-death process satisfying (20), that is,

$$\sum_{n=1}^{\infty} \frac{1}{\pi_n \lambda_n} = \infty.$$

we have

$$(4S_0)^{-1} \leq \lambda \leq S_0^{-1},$$

and hence

$$\lambda > 0 \quad if\ and\ only\ if \quad S_0 < \infty.$$

Further research

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- We will obtain some formulae for the values of the decay parameter in general Markov processes.

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