The decay parameter and the smallest Dirichlet eigenvalue of a birth-death process

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and Statistics of Complex Systems

Remark

This is joint work with Phil Pollett, University of Queensland.

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Some definitions

Definition Let E be a countable set, to be called the *state space*. A function $P_{ij}(t)$, $i, j \in E$, $t \geq 0$, is called a *transition function* on E if

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- $\sum_{j \in E} P_{ij}(t) \leq 1$ for all $t \geq 0, i \in E$.
- $P_{ij}(s+t) = \sum_{k \in E} P_{ik}(s) P_{kj}(t)$ for all $s, t \geq 0$ and $i, j \in E$ (this is called the Chapman-Kolmogorov equation, or semigroup property).

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- *standard* if $\lim_{t\to 0} P_{ii}(t) = 1$ for all $i \in E$;
- *honest* if $\sum_{j\in E} P_{ij}(t) = 1$ for all $t \geq 0$, $i \in E$, and *dishonest* otherwise.

Some definitions

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- Furthermore, the probability of any event involving at most countably many random variables $X(t_i)$ can be computed knowing only the transition function and initial distribution of the chain.
- All the probabilistic information about the process, insofar as it concerns only countably many time instants, is contained in the transition function and initial distribution. One could almost say that the transition function *is* the Markov Chain.

Some definitions

It is well known that

$$P'(0) = Q$$

exists in that

$$q_{ij} := \lim_{t \to 0} t^{-1} P_{ij}(t) \text{ exists in } [0, \infty) \text{ when } j \neq i.$$

$$q_i := -q_{ii} := \lim_{t \to 0} t^{-1} [1 - P_{ii}(t)] \text{ exists in } [0, \infty] \text{ for every } i.$$

Fatou's lemma shows that

$$\sum_{j \neq i} q_{ij} \leq q_i.$$

We then call $Q = (q_{ij}, i, j \in E)$ a q-matrix.

The decay parameter

Suppose we have a q-matrix Q over E. Let P be an arbitrary Q-transition function. Suppose that $E = \{0\} \cup C$, where 0 is an absorbing state and $C = \{1, 2, \cdots\}$ is irreducible. The decay parameter λ_C is defined by

$$\lim_{t \to \infty} -\frac{1}{t} \log P_{ij}(t) = \lambda_C.$$

Kingman showed that this limit exists and is the same for all $i, j \in C$, and that $0 \leq \lambda_C < \infty$.

The decay parameter

It is called the **decay parameter** because there exist constants $M_{ij} > 0$ with $M_{ii} = 1$ such that

$$P_{ij}(t) \leq M_{ij} e^{-\lambda_C t}, \qquad i, j \in C.$$

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If $\mu > \lambda_C$, there does not exist any μ-invariant measure (Pollett 1986); in particularly, there does not exist any **quasi-stationary distribution** if $\lambda_C = 0$.

The smallest Dirichlet eigenvalue

Now we shall start with discussing the general continuous-time Markov chain $(P_{ij}(t),\ i,j \in E,\ t \geq 0)$. Let $C \subset E$ and m is some probability measure in C such that

$$\sum_{i \in C} m_i P_{ij}(t) \leq m_j, \quad j \in C. \tag{1}$$

Let $L^2(m)$ be the Hilbert space of functions on E with

$$\|f\|^2 := \sum_{i \in E} f_i^2 m_i < \infty.$$

Here, and in what follows, we use convention $m_i = 0$, $i \in E \setminus C$.

The smallest Dirichlet eigenvalue

Let $\{P(t)\}_{t\geq 0}$ be a positive, strongly continuous, contractive and Markovian semigroup (i.e., $P(t)1 \leq 1 \; \forall t \geq 0$) on $L^2(m)$ induced by $(P_{ij}(t),\; i,j \in E,\; t \geq 0)$.

Denote by L and $\mathcal{D}(L)$ respectively the infinitesimal generator and its domain induced by $\{P(t)\}_{t\geq 0}$.

Let $A = E \setminus C$. We say that $P(t)$ is *exponentially ergodic* on A in the $L^2(m)$ norm if there is a positive α such that

$$\|P(t)f\| \leq e^{-\alpha t}\|f\|, \qquad f|_A = 0,\; \forall t \geq 0, \tag{2}$$

and

$$P_{ij}(t) = 0, \;\; i \in A,\; j \in E,\; \forall t \geq 0. \tag{3}$$

The smallest Dirichlet eigenvalue

One may define

$$\text{gap}(L) = \inf\{-(Lf, f):\ f \in \mathcal{D}(L),\ f|_A = 0,\ \|f\| = 1\}. \tag{4}$$

Our first step is to show that (2) and (4) are closely linked. To do so, let $D(f)$ denote the limit

$$\lim_{t\to 0} \frac{1}{t}(f - P(t)f, f) \geq \lim_{t\to 0} \frac{1}{2t} \sum_{i\in E} m_i (P(t)[f - f_i]^2)(i) \geq 0,$$

provided the limit exists. Next, define

$$\text{gap}(D) = \inf\{D(f):\ f \in \mathcal{D}(D),\ f|_A = 0,\ \|f\| = 1\}. \tag{5}$$

The smallest Dirichlet eigenvalue

Finally, define

$$\delta(t) = -\sup\{\log \|P(t)f\| : \; f|_A = 0, \; \|f\| = 1\}. \qquad (6)$$

Since

$$\|P(t+s)f\| \le e^{-\delta(t)}\|P(s)f\| \le e^{-\delta(t)-\delta(s)}\|f\|$$

by the contraction and semigroup properties, it follows that $\delta(t)$ is superadditive and $\delta(0) = 0$. Hence, the limit

$$\delta := \lim_{t\to 0}\frac{\delta(t)}{t} = \inf_{t>0}\frac{\delta(t)}{t} \qquad (7)$$

is well defined.

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Theorem 1 $\delta = \text{gap}(D) = \text{gap}(L)$.

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Theorem 2 $\text{gap}(D) \leq \lambda_C$.

The smallest Dirichlet eigenvalue

Three books on the relationship between Dirichlet forms and Markov processes.

1. N. Boulean and F. Hirsch, *Dirichlet Forms and Analysis on Wiener Space*, Walter de Gruyter, Berlin, New York, 1991.
2. Z.M. Ma, M. Röckner, *An Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*, Springer, 1992.
3. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov process*, Walter de Gruyter, Berlin, New York, 1994.

The birth-death process

We shall adopt the usual notation in prescribing birth rates $\lambda_i > 0 \ (i \geq 1)$, with $\lambda_0 = 0$, and death rates $\mu_i > 0 \ (i \geq 1)$ on E. Now define by $\pi_1 = 1$ and

$$\pi_n = \prod_{k=2}^{n} \frac{\lambda_{k-1}}{\mu_k}, \qquad n \geq 2.$$

We will assume the process is absorbed with probability 1, that is,

$$\sum_{n=1}^{\infty} \frac{1}{\pi_n \lambda_n} = \infty. \tag{8}$$

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- to give criteria for decay parameter λ_C to be positive in terms of the rates $(\lambda_n, \mu_n)_{n\geq 1}$;
- to determine the value of λ_C, or at least bounds for λ_C, in terms of the rates.

These are open problems. Many authors worked on them, for example, Callaert (1971), (1974), Callaert and Keilson(1973a),(1973b) and Van Doorn (1985), (1991).

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In order to state our main results, we need the following notation:

$$Q_n = \left(\frac{1}{\pi_1 \mu_1} + \sum_{j=1}^{n-1} \frac{1}{\pi_j \lambda_j} \right) \sum_{j=n}^{\infty} \pi_j, \quad n \geq 1,$$

and

$$S_0 = \sup_{n \geq 1} Q_n.$$

The birth-death process

Theorem 4 $(4S_0)^{-1} \le \lambda_C \le S_0^{-1}$.

And, hence,

$$\lambda_C > 0 \quad if\ and\ only\ if \quad S_0 < \infty.$$

The birth-death process

Let $\{X(t), t \geq 0\}$ be the birth-death process with $P_{ij}(t)$ as its transition function.

The first-passage (hitting) time into state 0 is defined by

$$\tau_0 = \inf\{t \geq 0 : X(t) = 0\}$$

($\tau_0 = 0$ if $X(0) = 0$).

We denote the moment generating function of τ_0, given $X(0) = i$, by

$$G_i(\alpha) = E_i[e^{\alpha\tau_0}].$$

It can be shown that the abscissa of convergence of $G_i(\alpha)$ is the same for all $i > 0$; we denote this by α^*.

The birth-death process

Theorem 5 $\lambda_C = \alpha^*$.

In order to prove this theorem, we use the stochastic monotonicity of birth-death process and a very useful method from modern probability—Coupling. I will not give details here.

Conclusion

- The decay parameter plays an important role in studying properties of Markov chains. It has a close relationship with the important concept of the smallest Dirichlet eigenvalue. They are also linked to another important concept: the abscissa of convergence of the moment generating function of the hitting time.

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- For a general Markov Chain, we have

$$\delta = \text{gap}(D) = \text{gap}(L) \leq \lambda_C.$$

Conclusion

- For every birth-death process satisfying (8), we have

$$\text{gap}(D) = \lambda_C = \alpha^*$$

and

$$(4S_0)^{-1} \le \lambda_C \le S_0^{-1},$$

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Further research

- We will deal with the decay parameter in the more general case, for example, symmetric Markov chains.

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- We will discuss the relationship between the decay parameter and the smallest Dirichlet eigenvalue in general Markov processes.

Acknowledgement

I would like to acknowledge the support of the Australian Research Council Centre of Excellence for Mathematics and Statistics of Complex Systems.

Thanks are due to Phil Pollett , Ben Cairns and David Sirl.