Metapopulation processes in stochastic environments

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Habitat dynamics

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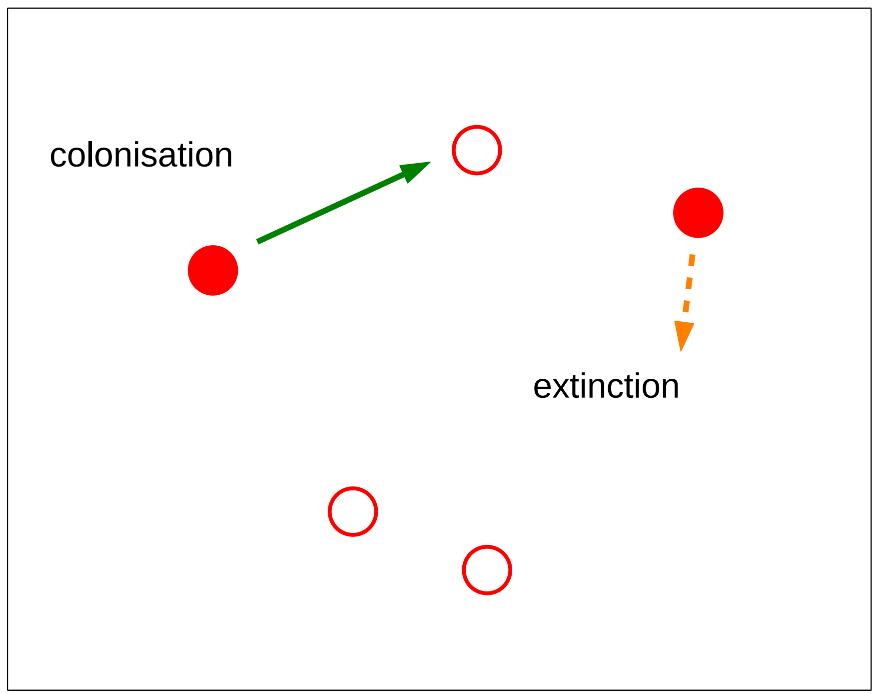

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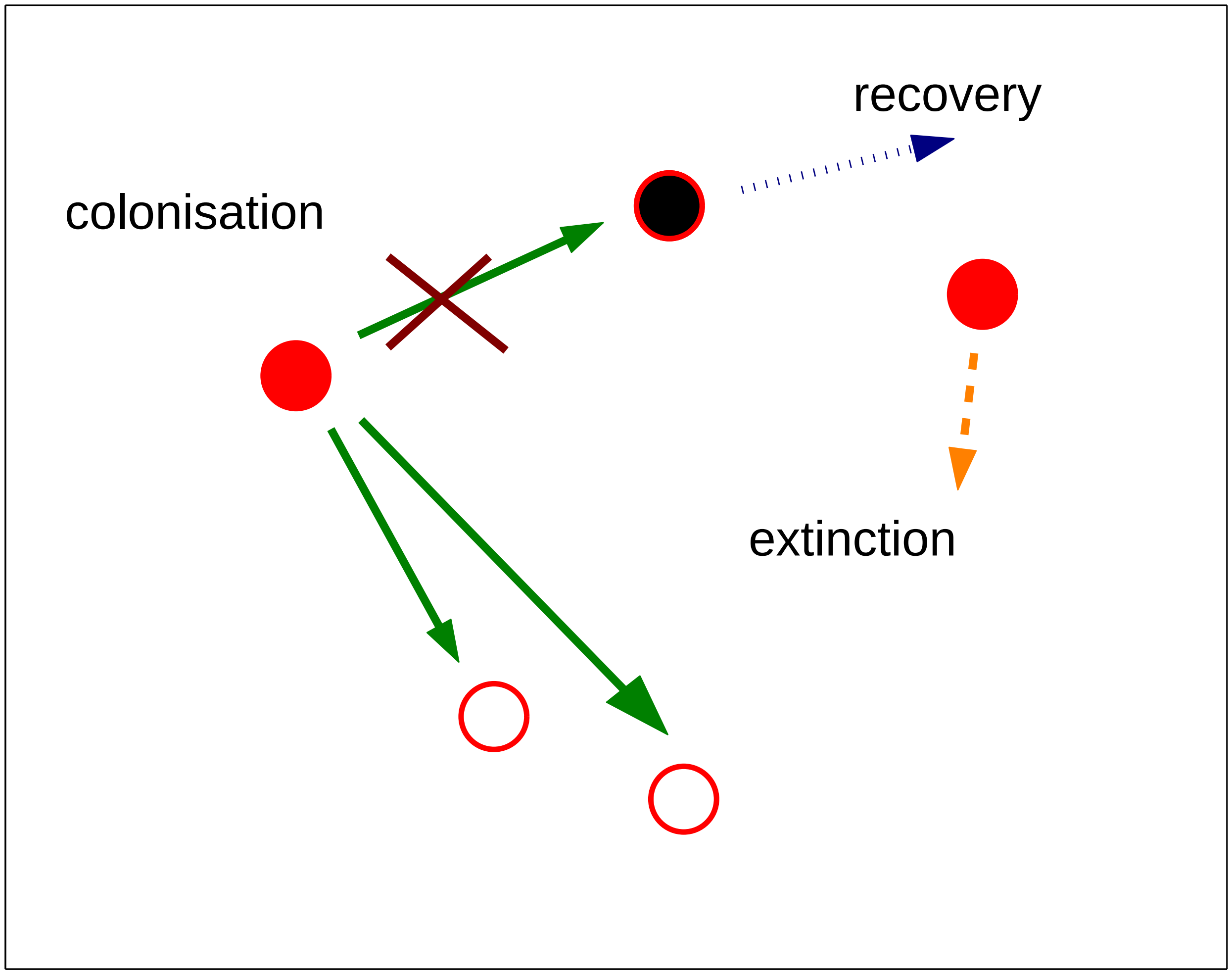
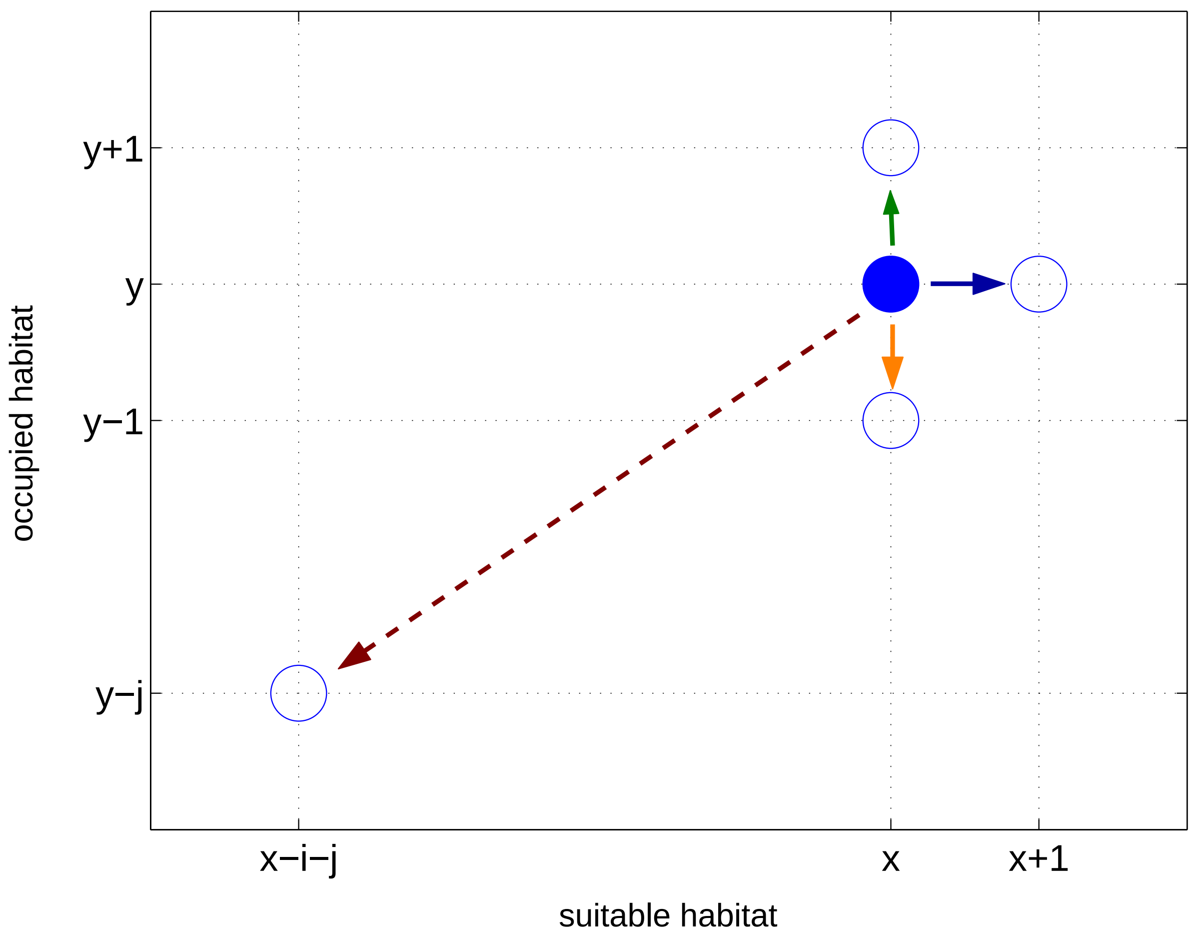
  - Habitat may be affected by environmental events...

A metapopulation with habitat dynamics



Transition rates...

For $(x, y) \in \mathbb{N}^2$, such that $0 \leq y \leq x \leq N$,



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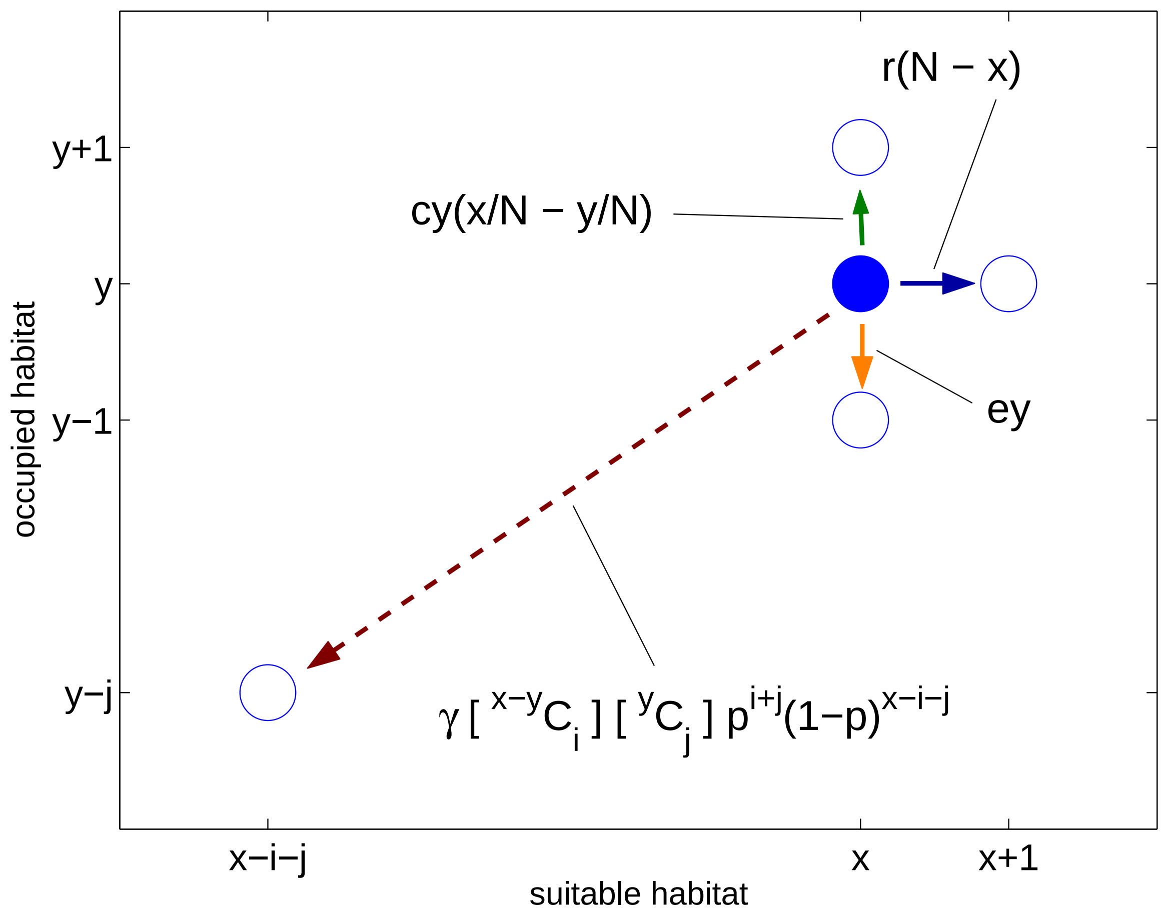

Difficulties as $N \uparrow$

When N is finite, we can hope to evaluate measures of interest (hitting times, QSDs, etc) directly.

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- To make progress, we need good approximations: *e.g.* stochastic differential equations for the limit as $N \to \infty$?

Simulations

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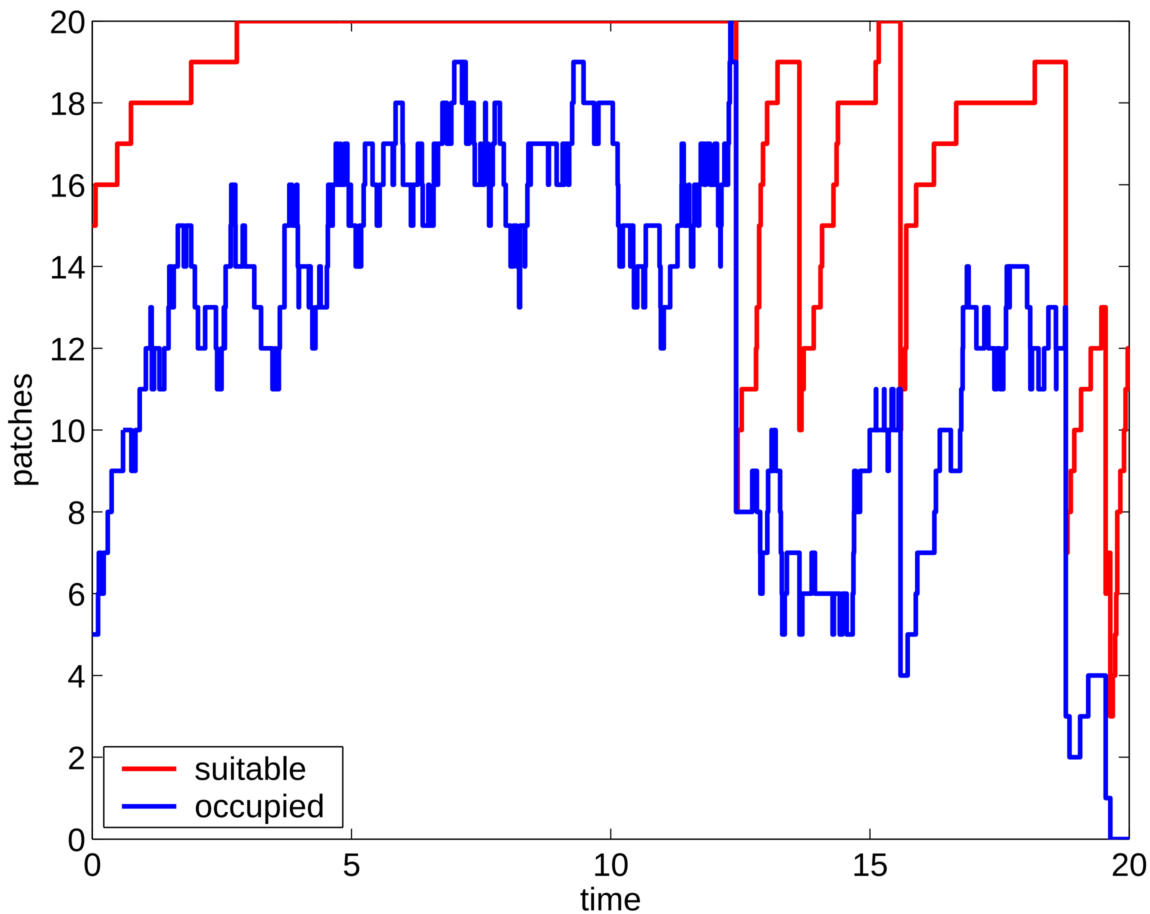

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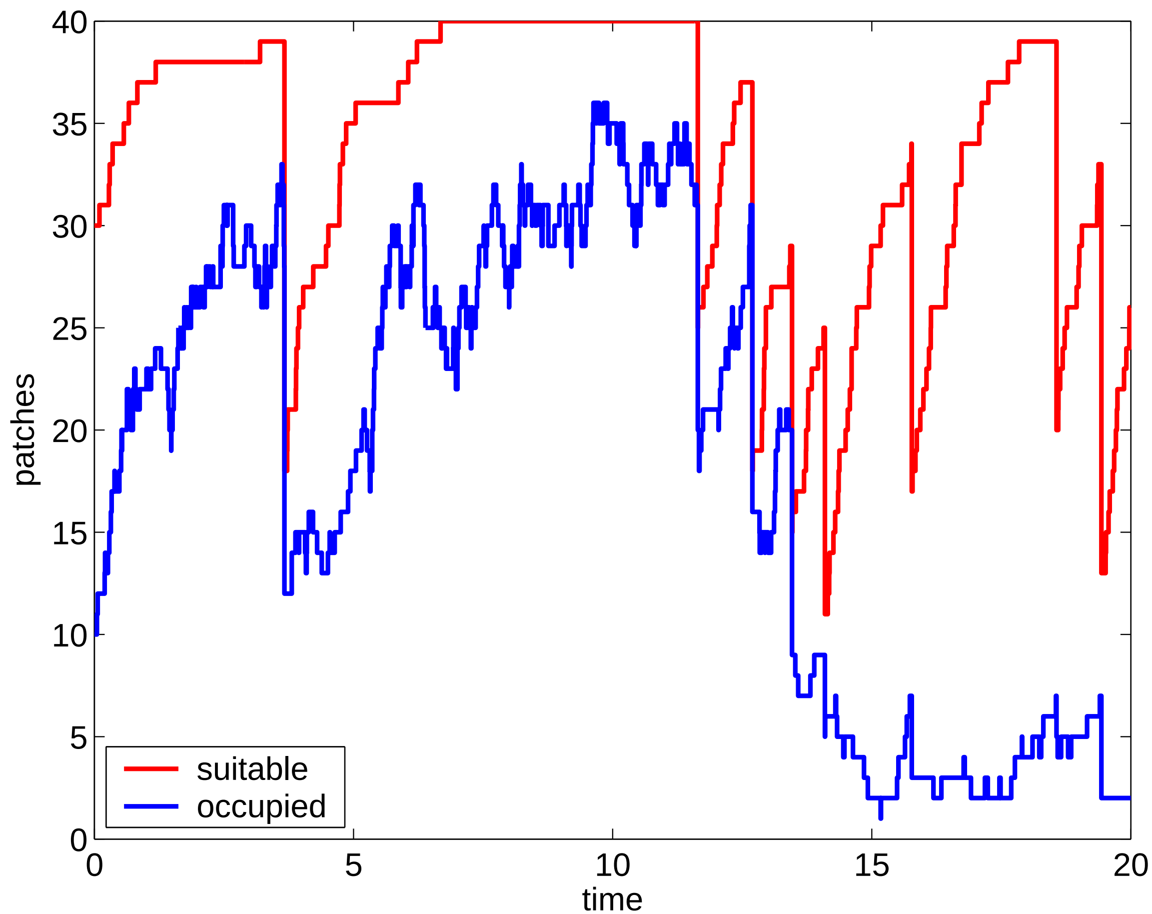

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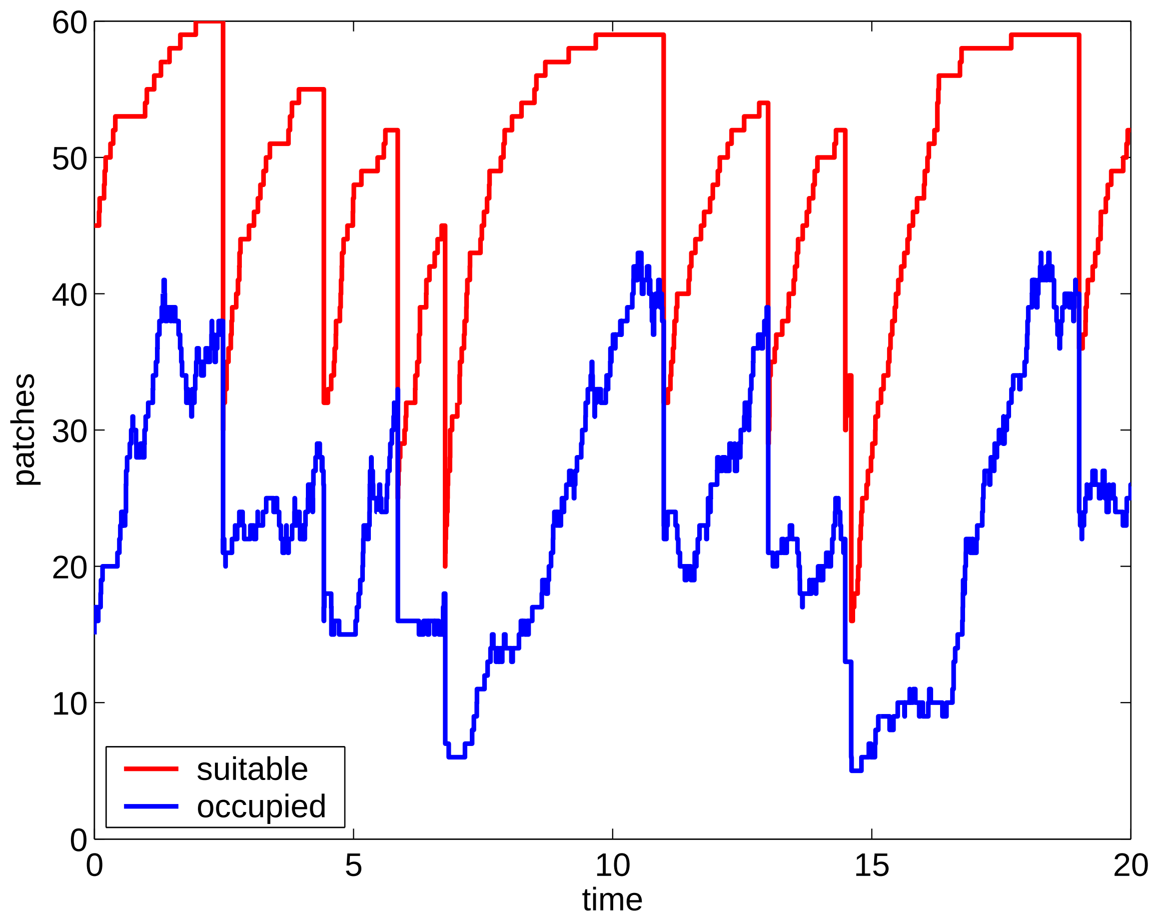

Interlude: (i) density dependent processes

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Strong law of large numbers. If $\{X_n\}$ is a density dependent family, if $F(x) = \sum_j j\beta_j(x)$, and $X(t) = x_0 + \int_0^t F(X(s))dt$, then $\sup_{s\leq t}|X_n(s) - X(s)| \to 0$, almost surely (under mild conditions on the β_j's and F).

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Central limit theorem. If $\{X_n\}$ and X are as above, and if $V_n = \sqrt{n}(X_n - X)$, then if $V_n(0) \to V(0)$, $V_n \Rightarrow V$ where V is a particular diffusion process with drift (under additional continuity conditions).

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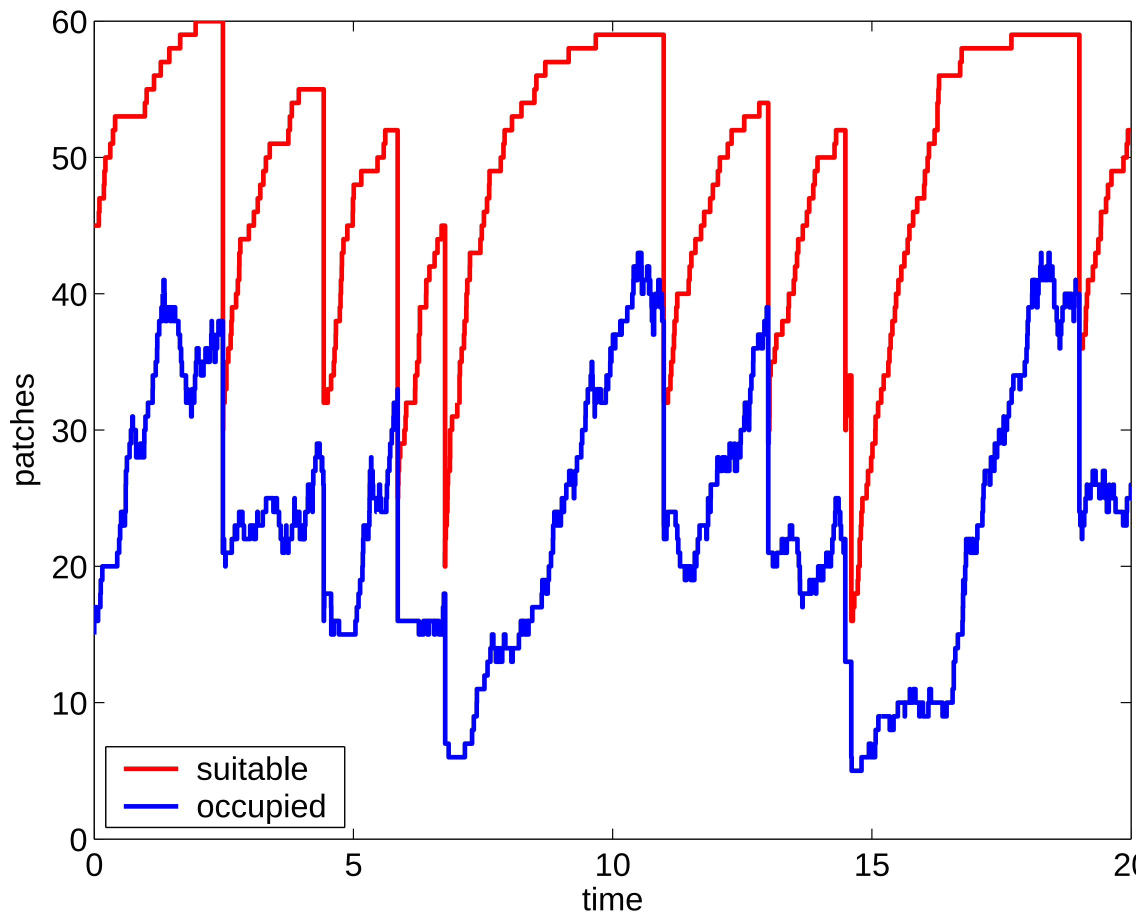

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Define $\hat{\mathbf{X}}$ as the piecewise deterministic process:

- deterministic between catastrophes (SLLN);
- catastrophes occur according to Poisson arrivals, but with size $p\hat{\mathbf{X}}$.

A central limit scaling (ii)

Define $\mathbf{Z}_N = \sqrt{N}(\mathbf{X}_N/N - \hat{\mathbf{X}})$, where we assume that catastrophes in $\mathbf{X}_N$ and $\hat{\mathbf{X}}$ occur at arrival times of the same Poisson process.

Then, we hope

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What is the form of $\mathbf{Z}$?

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The generator of $[\mathbf{Z}_N^\top \ \hat{\mathbf{X}}^\top]^\top$, $G_N f \to G f$ as $N \to \infty$, where Gf has:

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- Catastrophe component obtained by $\mathrm{E}[f(Z_1 + U_1, Z_2 + U_2, \cdot) - f(Z_1, Z_2, \cdot)]$, where (U_1, U_2) is bivariate normal.

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We do this to establish (by theorems of Ethier & Kurtz, 1986) that $[\mathbf{Z}_N^\top \ \hat{\mathbf{X}}^\top]^\top \Rightarrow [\mathbf{Z}^\top \ \hat{\mathbf{X}}^\top]^\top$.

So what now? Hitting times.

If G is the generator of a diffusion process subject to jumps at Poisson arrival times and H is an open set, then the minimal, non-negative solution $h(x, y)$ to

$$Gh(x, y) = -1, \quad (x, y) \in H \subset S,$$
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Let G be the generator of $\hat{\mathbf{X}} + \sqrt{N}\mathbf{Z}$; we could approximate expected times to extinction of the metapopulation.

Future directions

- Extension of this approach to general density dependent processes subject to a wider class of catastrophes.
- Investigation of the accuracy of the approximations and their properties (e.g. hitting times).
- When do features of small-N processes remain in large-N approximations?

Thanks

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