Unbounded integrands

This file contains explanations for choices of gauges in the *Maple* worksheet examples.mw.

Example 1 The integrand is $f_1(x) = -\ln|x|$. In the worksheet the definition is extended at zero as zero. Let the expected value of the integral over an interval $[u, v]$ be $F(v) - F(u)$. Assume that the tagged division D of $[-1, 1]$ is d_1-fine, for some not yet specified d_1 which ensures that zero tags every interval which contains it. Let D_0 be the part of D which does not contain any interval tagged by zero. Then we have, for some $z \in [u, v]$,

$$W = \left|\sum_{D_0}(F(v) - F(u) - f(x)(v-u)\right|$$
$$\leq \left|\sum_{D_0}\left((v-u)\ln\frac{z}{x}\right)\right|$$
$$\leq \sum_{D_0}(v-u)\ln\frac{|x| + d_1(x)}{|x| - d_1(x)}.$$

If we can choose d_1 in such a way that

$$\ln\frac{|x| + d_1(x)}{|x| - d_1(x)} \leq \varepsilon, \tag{1}$$

then $W \leq 2\varepsilon$. Luckily, it is possible to satisfy the inequality (1) by defining $(0 < \varepsilon < 1)$

$$d_1(x) = |x|\frac{e^\varepsilon - 1}{e^\varepsilon + 1} \quad \text{for} \quad x \neq 0.$$

This choice also forces zero to be the tag of the subinterval which contains it. If $d_1(0) = \varepsilon$ with $0 < \varepsilon < 1$ then we have altogether

$$\left|\sum_{D}(F(v) - F(u) - f(x)(v-u)\right| \leq 4\varepsilon - 2\varepsilon\ln\varepsilon.$$

For the display we have choosen $\varepsilon = 0.12$.

Example 2 The integrand is $f_2(x) = \frac{1}{\sqrt{|x|}}$. In the worksheet the definition is extended at zero to be 0. Let $F_2(x) = \kappa\sqrt{|x|}$ with $\kappa = 2$ for $x \geq 0$ and $\kappa = -2$ for $x < 0$. The expected value of the integral over an interval

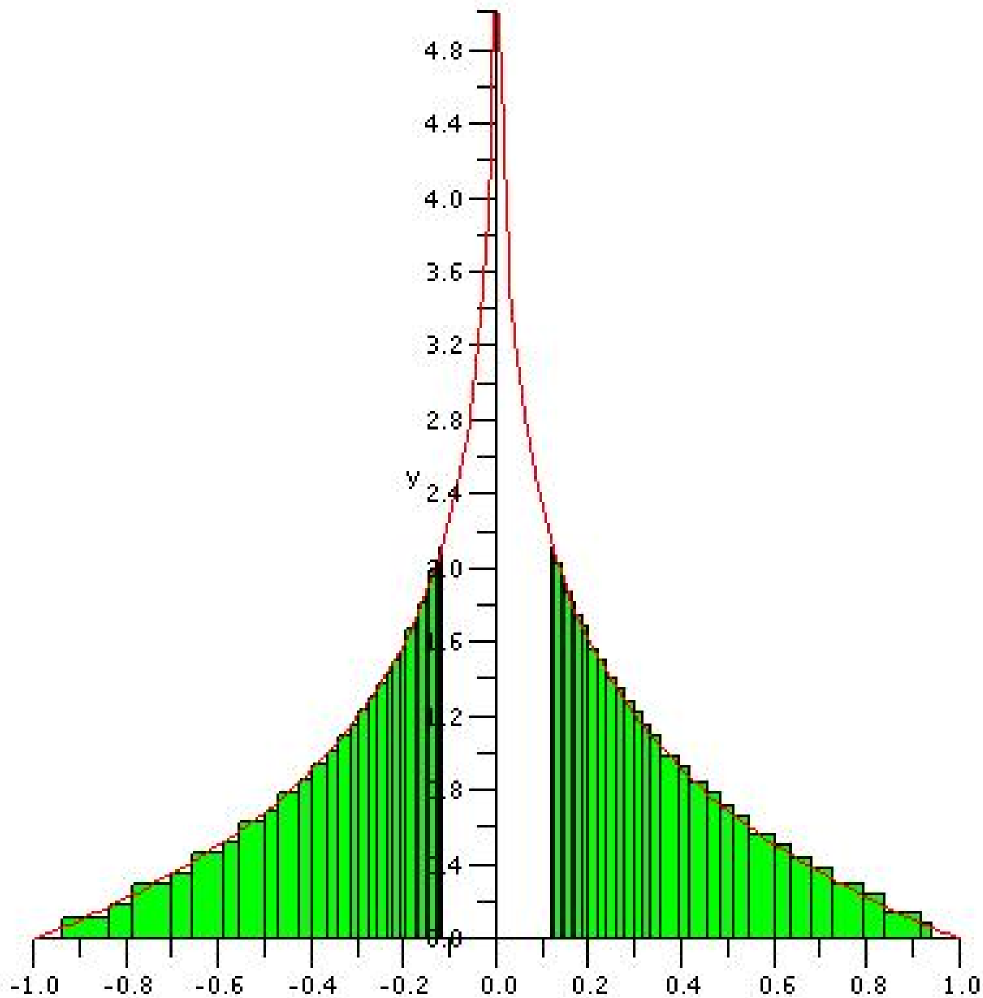

Figure 1: ln on $[-1, 1]$

$[u,\, v]$ is $F_2(v) - F_2(u)$. Assume that the tagged division of $[-1,1]$ is d_2-fine, for some not yet specified d_2. We consider the interval $[0,1]$ first. Then we have

$$
\begin{aligned}
W = &\left|\sum\left[F_2(v) - F_2(u) - \frac{1}{\sqrt{x}}(v-u)\right]\right| \\
&\leq \left|\sum \frac{2\sqrt{x} - (\sqrt{(v)} + \sqrt{(u)})}{\sqrt{x}}\right| (\sqrt{v} - \sqrt{u}) \\
&\leq \sum (\sqrt{v} - \sqrt{u}) \left(\sqrt{1 + \frac{d_2(x)}{x}} - \sqrt{1 - \frac{d_2(x)}{x}}\right).
\end{aligned}
$$

The choice $d_2 = \varepsilon_2 |x|$ with $0 < \varepsilon_2 < 1$ is now fairly obvious and we obtain

$$
W \leq \sqrt{1+\varepsilon_2} - \sqrt{1-\varepsilon_2} \leq 2\varepsilon_2.
$$

The choice of d_2 also forces zero to be the tag of the subinterval which contains it. If $d_2(0) = \varepsilon_2$ then we have altogether

$$
\left|\sum (F_2(v) - F_2(u) - f_2(x)(v-u)\right| < 2\varepsilon_2 + 2\sqrt{\varepsilon_2}. \tag{2}
$$

The resoning for the interval $[-1,0]$ is similar. For the display we have choosen $\varepsilon_2 = 0.1$.

Example 3 The basic interval is $[-1,1]$. The integrand f_3 is defined as follows

$$
f_3(x) = \begin{cases} 0.1 & \text{if } x \leq -1 \\ \dfrac{1}{\sqrt{1-x^2}} & \text{if } x < 1 \\ 0.1 & \text{if } x \geq 1 \end{cases}
$$

Inspired by the previous examples we choose the gauge to be an ε_3 multiple of the distance to the points where the integrand is unbounded. More precisely

$$
d_3(x) = \begin{cases} \varepsilon_3 & \text{if } |x| = 1 \\ \varepsilon_3(1-|x|) & \text{if } |x| < 1 \end{cases}
$$

Assume that the tagged division of $[-1,1]$ is d_3-fine, with $0 < \varepsilon_3 < 1$. Then

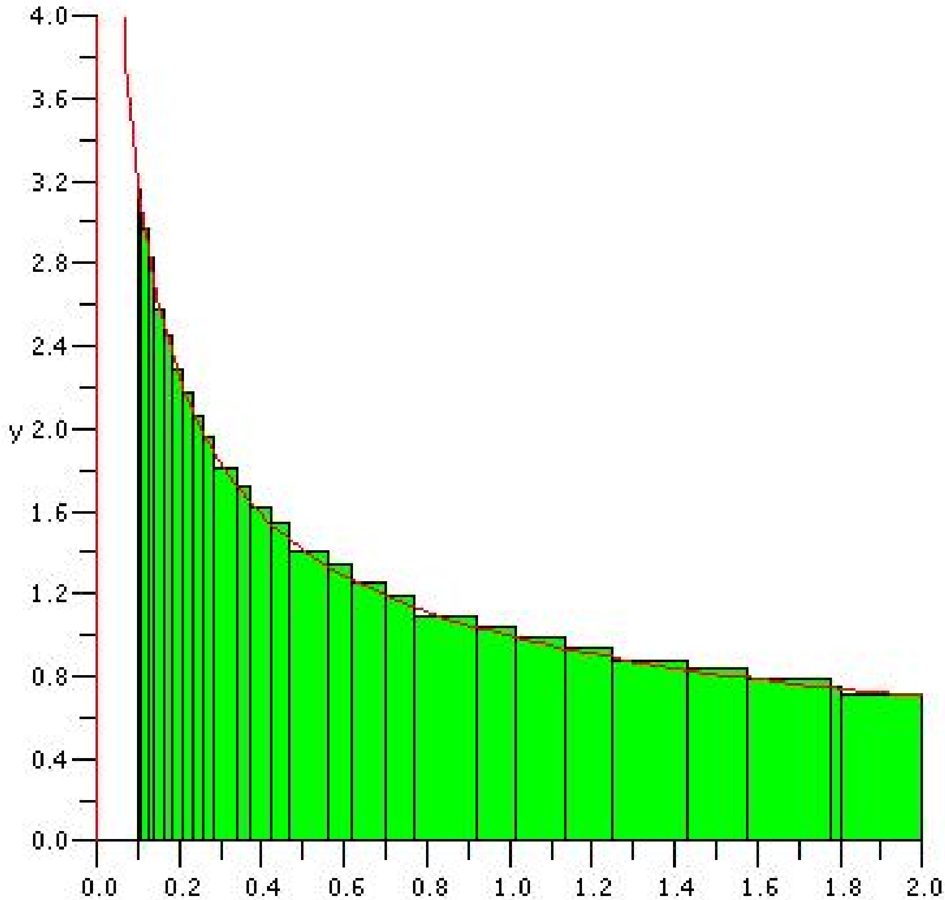

Figure 2: $(\sqrt{x})^{-1}$ on $[0, 1]$

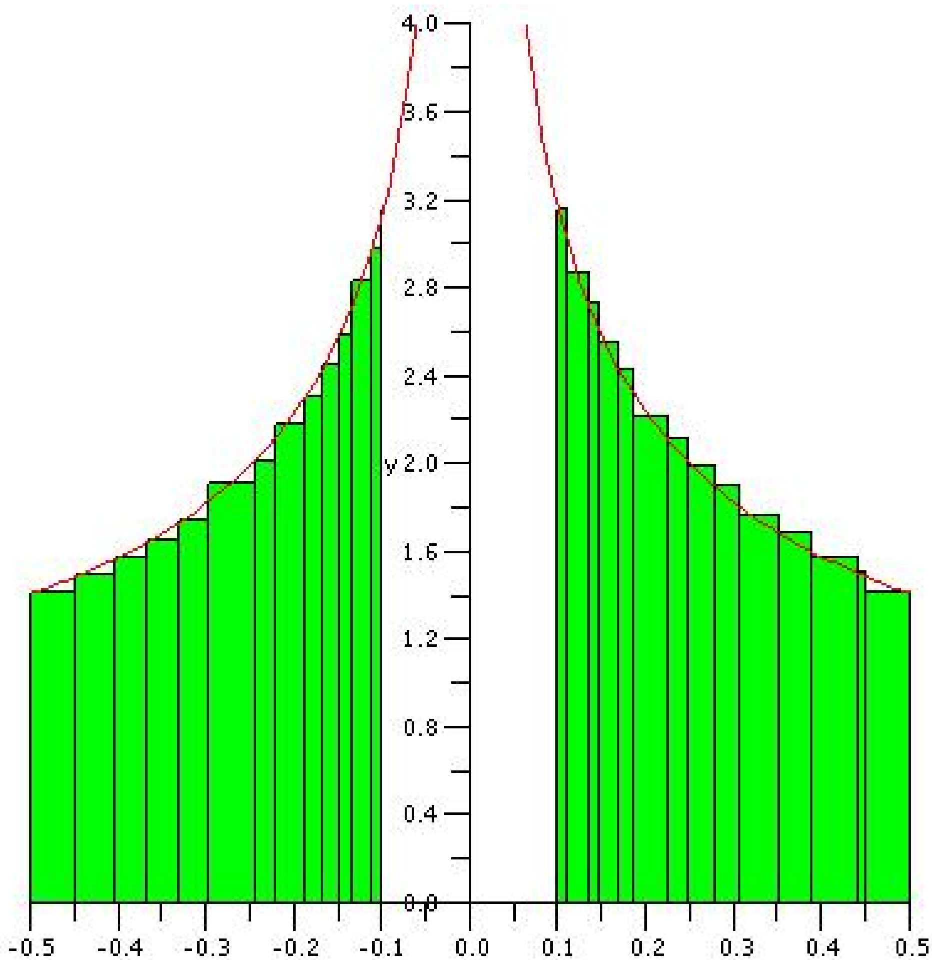

Figure 3: $\left(\sqrt{|x|}\right)^{-1}$ on $[-1, 1]$

we have, for some z with $u \le z \le v$,

$$W = \left| \sum_{-1<u,v<1} \left(\arcsin(v) - \arcsin(u) - \frac{v-u}{\sqrt{1-x^2}} \right) \right|$$
$$\le \sum_{-1<u,v<1} (\arcsin(v) - \arcsin(u)) \left| 1 - \sqrt{\frac{1-z^2}{1-x^2}} \right|.$$

The absolute value in the sum can be estimated as follows (noting that $|1-\sqrt{X}| \le |1-X|$ for $X > 0$)

$$\left| 1 - \sqrt{\frac{1-z^2}{1-x^2}} \right| \le \left| 1 - \frac{1-z^2}{1-x^2} \right|$$
$$\le \frac{|z^2-x^2|}{1-x^2} \le \frac{|z-x|}{1-|x|} \le \frac{d_3(x)}{1-|x|} \le \varepsilon_3.$$

This together with the choises for d_3 at ± 1 gives

$$\left| \sum (\arcsin(v) - \arcsin(u) - f_3(x)(v-u)) \right|$$
$$\le 2\varepsilon_3 \arcsin(1) + 2 \left(\arcsin(1) - \arcsin(1-\varepsilon_3) \right).$$

This proves that $\int_{-1}^{1} f_3 = 2\arcsin(1)$.

For the display we have chosen $\varepsilon_3 = 0.09$.

Example 4 The integrand f_4 is defined as follows:

$$f_4(x) = \begin{cases} \dfrac{1-\lfloor x \rfloor}{\sqrt{1-|x|}} - \dfrac{1}{\sqrt{|x|}} & \text{if } x(1-|x|) \ne 0 \\ 0 & \text{if } x(1-|x|) = 0. \end{cases}$$

Let F_2 be defined as in Example 2. Let

$$G(x) = \begin{cases} 4\sqrt{1+x} & \text{if } -1 \le x \le 0 \\ -2\sqrt{1-x} + 6 & \text{if } 0 < x \le 1. \end{cases}$$

Note that G is continuous at 0. If $F_4 = G - F_2$ then we expect $\int_u^v f_4 = F_4(v) - F_4(u)$. Similarly as in the previous examples we choose the gauge

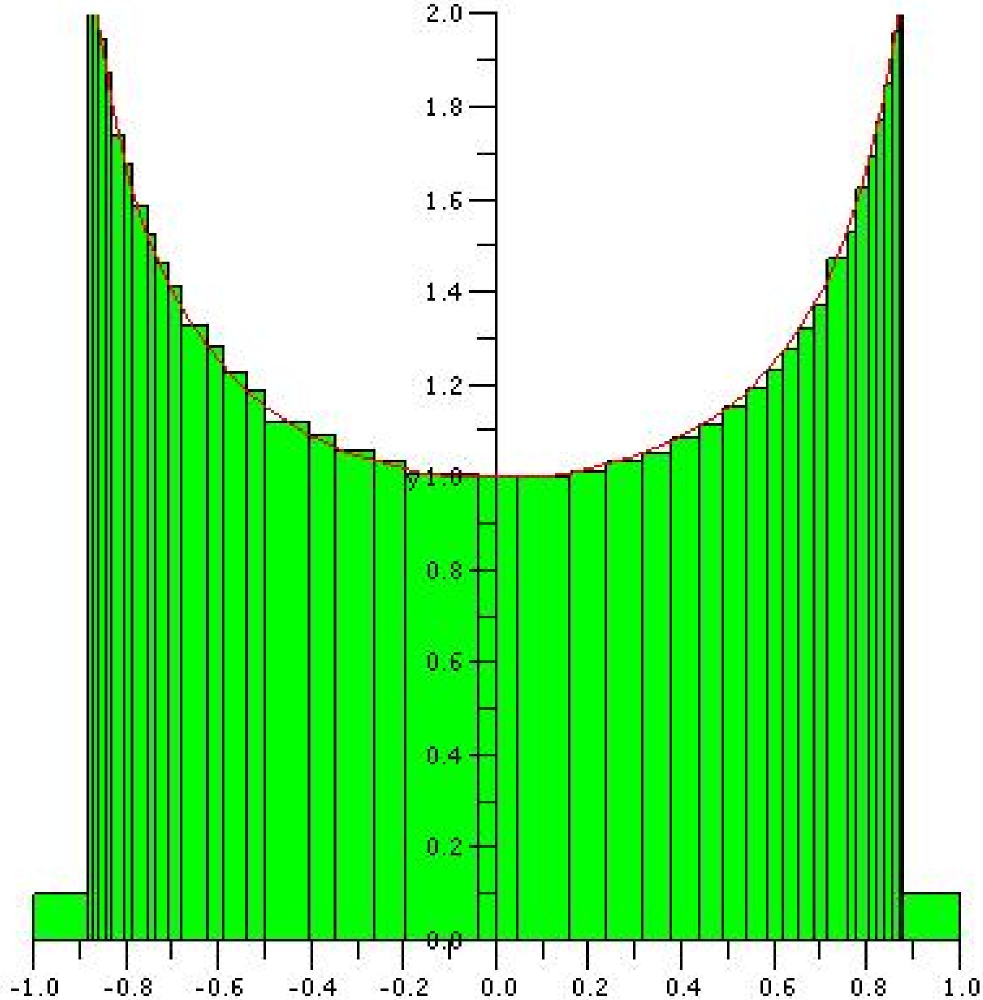

Figure 4: Singularities at both ends

proportional to the distance to the points in neighborhood of which the integrand is not bounded, more precisely

$$d_4(x) = \varepsilon_4 \left(\mathrm{Min}(1-|x|,|x|) + \lfloor 1+x \rfloor - \lceil x \rceil\right)$$

Let D be d_4-fine tagged division of $[-1,1]$. We break the sum

$$W = \sum_D \left(F_4(v) - F_4(u) - f_4(x)(v-u)\right)| \tag{3}$$

into several parts. Firstly we consider the tagged partial division D_0 in which the subintervals are tagged by $-1, 0, 1$. Let $[-1, V_{-1}]$ and $[U_1, 1]$ be the intervals tagged by -1 and 1, respectively. Denote by $[U_0, V_0]$ the interval or the union of intervals (if they are two) tagged by 0. Then we have

$$\begin{aligned}
&\left|\sum_{D_0} \left(F_4(v) - F_4(u) - f_4(x)(v-u)\right)\right| \le \left|\sum_{D_0} \left(F_4(v) - F_4(u)\right)\right| \\
&\le 4\sqrt{1+V_{-1}} + 2\left(1 - \sqrt{|V_{-1}|}\right) + \Big[G(V_0) - G(U_0)\Big] + \Big[F_2(V_0) - F_2(U_0)\Big] \\
&\qquad + 2\sqrt{1-U_1} + 2\left(1 - \sqrt{U_1}\right) \\
&\le 4\sqrt{\varepsilon_4} + 2\varepsilon_4 + 6\left(\sqrt{1+\varepsilon_4} - 1\right) + 4\varepsilon_4 + 2\sqrt{\varepsilon_4} + 2\varepsilon_4 \le 11\varepsilon_4 + 6\sqrt{\varepsilon_4}
\end{aligned} \tag{4}$$

Denote by $D_\pm$ that part of D in which the tags are not equal to -1 or 0 or 1. Similarly as in Example 2. we have

$$\left|\sum_{D\pm} F_2(v) - F_2(u) - f_2(x)(v-u)\right| \le 4(\varepsilon_4 + \sqrt{\varepsilon_4}). \tag{5}$$

We obtain for some $z \in [u,v]$

$$\left|G(v) - G(u) - \frac{1 - \lfloor x \rfloor}{\sqrt{1-|x|}}(v-u)\right| \le [G(v) - G(u)] \left|1 - \frac{\sqrt{1-|z|}}{\sqrt{1-|x|}}\right|. \tag{6}$$

Since

$$\left|1 - \frac{\sqrt{1-|z|}}{\sqrt{1-|x|}}\right| \le \left|\frac{|x|-|z|}{1-|x|}\right| \le \varepsilon_4, \tag{7}$$

we have by inequalities (6) and (7)

$$\left|\sum_{D\pm}(G(v)-G(u)-g(x)(v-u))\right| \leq G(1)\varepsilon_4 = 6\varepsilon_4. \tag{8}$$

Combining inequalities (5) and (8) leads to

$$\left|\sum_{D\pm} F_4(v)-F_4(u)-f_4(x)(v-u)\right| \leq 10\varepsilon_4 + 4\sqrt{\varepsilon_4}.$$

Finaly we have by (4) and the last inequality

$$\left|\sum_{D} F_4(v)-F_4(u)-f_4(x)(v-u)\right| \leq 21\varepsilon_4 + 10\sqrt{\varepsilon_4}.$$

For the display $\varepsilon_4 = 0.1$

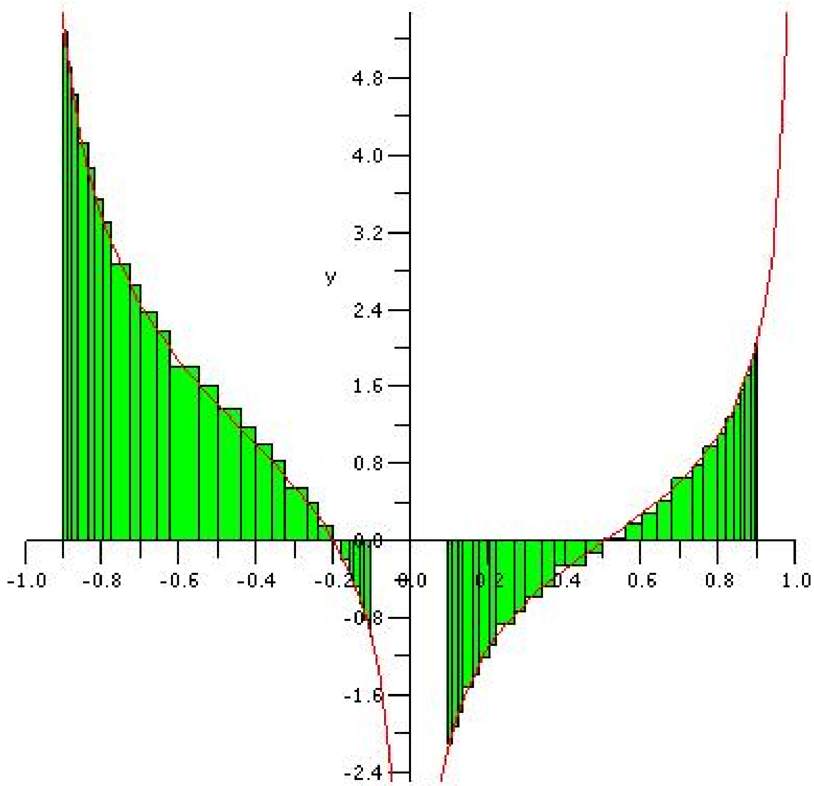

Figure 5: Singularities at both ends and in the centre