Further decompositions of complete tripartite graphs into 5-cycles

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Abstract

Let $K(r, s, t)$ denote the complete tripartite graph with partite sets of size r, s and t, where $r \leq s \leq t$. Necessary and sufficient conditions are given for decomposability of $K(r, s, t)$ into 5-cycles whenever r, s and t are all even. This extends work done by Mahmoodian and Mirzakhani (1995) and Cavenagh and Billington (2000).

Key words: Graph decomposition, complete tripartite graph, 5-cycle, trade.

1 Introduction

A graph with vertex set V is said to be a *complete n-partite* graph if V may be partitioned into n disjoint non-empty sets $V_1, V_2, \ldots, V_n$ (called *partite sets*) such that there exists exactly one edge between vertices from different partite sets, and no other edges. If $|V_i| = a_i$ for $1 \leq i \leq n$, this graph is denoted by $K(a_1, a_2, \ldots, a_n)$. A *k-cycle*, with k edges $x_i x_{i+1}$, $1 \leq i \leq k-1$, and $x_k x_1$, on k distinct vertices x_i, $1 \leq i \leq k$, is denoted by $(x_1, x_2, \ldots, x_k)$ or $(x_1, x_k, x_{k-1}, \ldots, x_3, x_2)$ or any cyclic shift of these.

The problem of finding necessary and sufficient conditions to decompose complete n-partite graphs into k-cycles has been considered for many values of n and k. The case $n = 2$ was completely solved by Sotteau [8]. Clearly the case $n = 2$ forces the cycle length to be even; $n = 3$ is the smallest value which permits odd cycle length.

[1] Work based on part of M.Sc. Thesis [3].

It has been shown that a complete n-partite graph with each partite set of size k decomposes into k-cycles if and only if both n and k are odd [5]. Necessary and sufficient conditions to decompose the same graph into hamiltonian cycles are given in [6]. In the case of complete tripartite graphs, Cavenagh [2] showed that $K(m, m, m)$ can be decomposed into k-cycles if and only if $k \leq 3m$ and k divides $3m^2$. Billington [1] gave necessary and sufficient conditions for existence of a decomposition of any complete tripartite graph into specified numbers of 3-cycles and 4-cycles; the techniques used in that paper are extended and applied here.

The problem of decomposing complete tripartite graphs into 5-cycles was first considered by Mahmoodian and Mirzakhani [7] and further results were given by Cavenagh and Billington [4]. In this paper we give necessary and sufficient conditions for the decomposition of the complete tripartite graph $K(r, s, t)$ into 5-cycles for the case when all partite sets have even size.

To solve this case we exploit the machinery first constructed in [4]. In Section 3 we show a way of representing a complete tripartite graph that allows us to monitor *trades* of sets of triangles and other edges with 5-cycles. In Section 4 we classify these trades into various types, which are then used in Section 5 to give the required decompositions.

2 Necessary conditions

Here we give some necessary conditions for the decomposition of $K(r, s, t)$ into 5-cycles, and develop some theorems for later use. The following result is from Mahmoodian and Mirzakhani [7].

Theorem 1 *If the complete tripartite graph $K(r, s, t)$ (where $r \leq s \leq t$) can be decomposed into 5-cycles, then the following conditions are satisfied:*

(i) r, s *and* t *are either all even or all odd;*
(ii) $5|(rs + rt + st)$;
(iii) $t \leq 4rs/(r + s)$.

Proof Condition (i) arises from the fact that the degree of each vertex must be even for a decomposition to exist. Condition (ii) simply states that the number of edges must be divisible by five. To prove (iii), first observe that each 5-cycle must use at least one edge between any two partite sets, in particular the partite sets of smallest sizes r and s. Therefore rs, the number of edges between the two smallest partite sets, must be greater than or equal to the total number of 5-cycles, $(rs + st + rt)/5$. The result follows. □

Corollary 2 *If the complete tripartite graph $K(r,s,t)$ (with $r \le s \le t$) can be decomposed into 5-cycles, then $t \le 3r$, $s \le 3r$ and $t \le 2s$.*

Proof From condition (iii) in the previous theorem we have that $s \le t \le 4rs/(r+s)$, so that $(r+s) \le 4r$, from which we have $s \le 3r$. Therefore $t \le 4rs/(r+s) \le 4rs/(4s/3) = 3r$. Also, $t \le 4rs/(r+s) \le 4rs/(2r) = 2s$. □

Corollary 3 *If the complete tripartite graph $K(r,s,t)$ (with $r \le s \le t$) can be decomposed into 5-cycles, then at least two of the three partite sets have sizes which are congruent modulo 5.*

Proof Let $r \equiv r' \pmod 5$, $s \equiv s' \pmod 5$ and $t \equiv t' \pmod 5$, where $0 \le r', s', t' \le 4$. Then the triple (r', s', t') must belong to the following list:

$$(0,0,0),\ (0,0,1),\ (0,0,2),\ (0,0,3),\ (0,0,4),$$
$$(0,1,0),\ (0,2,0),\ (0,3,0),\ (0,4,0),\ (1,0,0),$$
$$(2,0,0),\ (3,0,0),\ (4,0,0),\ (1,1,2),\ (1,2,1),$$
$$(2,1,1),\ (1,3,3),\ (3,1,3),\ (3,3,1),\ (2,2,4),$$
$$(2,4,2),\ (4,2,2),\ (3,4,4),\ (4,3,4),\ (4,4,3).$$

This follows from the fact that $rs+rt+st$ is divisible by 5; it is straightforward to check that these are all the possible cases. □

The following theorem is the main result from Mahmoodian and Mirzakhani's paper [7].

Theorem 4 *Let $K(r,s,t)$ be a complete tripartite graph such that $r = s \le t$ or $r \le s = t$ and the conditions of Theorem 1 are satisfied. Then $K(r,s,t)$ has a decomposition into 5-cycles except possibly when two partite sets have order divisible by 5, but the third partite set does not.*

This result was extended by Cavenagh and Billington [4] who showed the following.

Theorem 5 *Let $K(r,s,t)$ be a complete tripartite graph satisfying the necessary conditions given in Theorem 1. Then $K(r,s,t)$ decomposes into 5-cycles if either two partite sets have the same size, or both r and s are divisible by 10.*

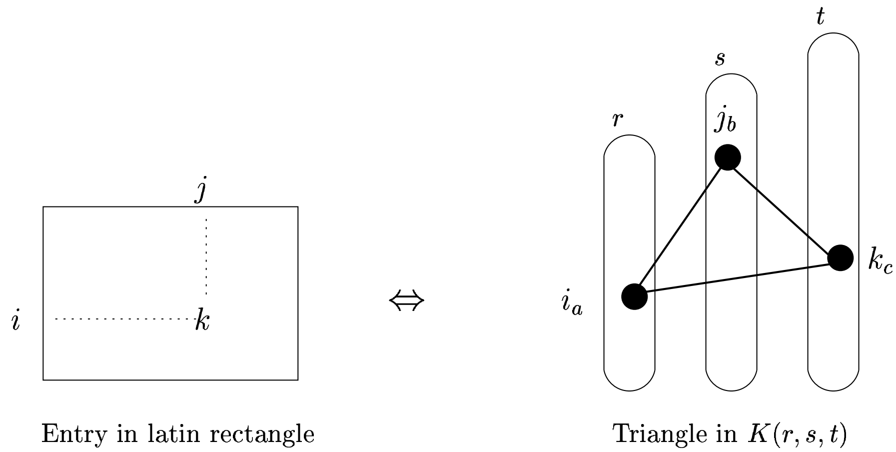


Fig. 1.

3 Latin representations

We now develop a way of representing a complete tripartite graph which in fact extends the idea of a latin square. This representation (first used in [4]) will be an important tool for constructing decompositions of complete tripartite graphs.

Definition 6 *Consider a rectangular array of integers of order $r \times s$ with entries from the set $T = \{1, 2, \ldots, t\}$. If each entry appears at most once in each row and at most once in each column, we call such an array a* latin rectangle *of order $r \times s$ based on t elements.*

It is well-known that a latin square of order m is equivalent to a decomposition of the complete tripartite graph $K(m, m, m)$ into triangles. An interesting extension of this result is the following lemma.

Lemma 7 *Let r, s and t be integers such that $r \leq s \leq t$. A latin rectangle of order $r \times s$ based on t elements is equivalent to the existence of rs edge-disjoint triangles sitting inside the complete tripartite graph $K(r, s, t)$.*

Proof First, label the vertices of $K(r, s, t)$ as follows:

$$\{1_a, 2_a, \ldots, r_a\} \cup \{1_b, 2_b, \ldots, s_b\} \cup \{1_c, 2_c, \ldots, t_c\}.$$

Then take a latin rectangle of order $r \times s$ based on t elements, with rows labelled from 1 to r and columns labelled from 1 to s. For each entry k in row i and column j we take a 3-cycle (i_a, j_b, k_c) in our tripartite graph. This can be seen more clearly in Figure 1.

Since no entry appears more than once in any row or column of a latin rectangle, the set of all such 3-cycles is edge-disjoint. $\square$

So an $r \times s$ latin rectangle based on t elements can be thought of as a representation of rs triangles in the complete partite graph $K(r, s, t)$. Of course not all edges of $K(r, s, t)$ are represented in these rs triangles (unless $r = s = t$). However we can in fact add extra entries to the rows and columns of a latin rectangle so that *every* edge in $K(r, s, t)$ is represented.

Definition 8 *Let r, s and t be integers such that $r \leq s \leq t$. A* latin representation *of the complete tripartite graph $K(r, s, t)$ is a latin rectangle of order $r \times s$ based on t elements, together with a set of $t-s$ entries at the end of each row and a set of $t-r$ entries at the bottom of every column* so that *each entry from the set $T = \{1, 2, \ldots, t\}$ occurs once in each of the r rows and once in each of the s columns.*

So to construct a latin representation of the complete tripartite graph $K(r, s, t)$ we first take a latin rectangle of order $r \times s$ based on t elements. We then adjoin to the end of each row any elements from the set T not already used in that row (in any order). Finally, to the bottom of each column we adjoin any entries from the set T not already used in that column.

Each entry at the end of a row represents a single edge from the partite set of size r to the partite set of size t. For example, the entry k at the end of row i represents the edge $\{i_a, k_c\}$. Similarly, the entry k at the bottom of column j represents the edge $\{j_b, k_c\}$. So a latin representation of $K(r, s, t)$ is in fact equivalent to a decomposition of $K(r, s, t)$ into rs triangles and $r(t-s)+s(t-r)$ single edges.

Example 9 Figure 2 is a latin representation of the graph $K(5, 5, 7)$. We may think of this as a decomposition of $K(5, 5, 7)$ into 25 triangles and 20 further edges. For clarification we always use a double line to separate entries within the latin rectangle from entries outside the latin rectangle. Moreover, we stress that each entry inside the latin rectangle represents a triangle (*three* edges), whereas each entry outside the latin rectangle represents a single edge.

Certain kinds of latin representations will prove invaluable in finding decompositions of the complete tripartite graph $K(r, s, t)$ into 5-cycles. Our idea is to *trade* sets of triangles and other edges in $K(r, s, t)$ with 5-cycles, using our latin representation to keep a record of such exchanges, until all edges of $K(r, s, t)$ are used. This is a slight variation on the normal idea of a trade (see [9] for instance), so for the purpose of this paper we make the following definition.

Definition 10 *Let M be a latin representation of the complete tripartite graph*

1	**2**	3	4	5	**6**	**7**
2	3	6	7	4	1	5
3	5	7	6	2	1	4
4	6	1	3	7	2	5
5	7	4	2	1	3	6
6	1	2	1	3		
7	4	5	5	6		

Fig. 2.

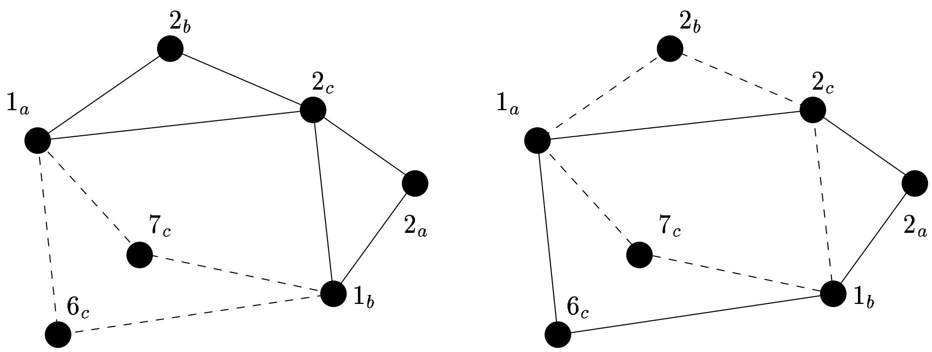


Fig. 3.

$K(r, s, t)$. *A* trade *is a set of entries in M, corresponding to a set of triangles and edges in $K(r, s, t)$ which can be decomposed into* 5*-cycles.*

Example 11 Consider the latin representation of the complete tripartite graph $K(5, 5, 7)$ given in Example 9. In Figure 2 six entries of this latin representation are in bold. These six entries correspond to the two triangles and four edges, shown on the left of Figure 3, the edges of which decompose into two 5-cycles, shown on the right of Figure 3. Thus these six entries in fact form what we refer to in this paper as a trade.

4 Classification of 5-cycle trades

In this section we find and classify a range of trades that are needed subsequently.

Consider a latin representation M (containing a latin rectangle L) of a complete tripartite graph $K(r, s, t)$. Recall that a *trade* is a set of entries in M, corresponding to edges in $K(r, s, t)$ which are decomposable into 5-cycles. We shall use two types of trades. In the first type, entries from both inside and outside our latin rectangle L are used. In this type of trade we are exchanging a set of triangles and a set of edges with a set of 5-cycles. In the second type, no entries from outside the latin rectangle are used. In this case we are trading a set of triangles with a set of 5-cycles.

We define a *relabelling* of the entries of a trade to be a bijection ϕ from the set of entries $T = \{1, 2, \ldots, t\}$ onto itself. Thus every occurrence of $i \in T$ in the trade is replaced by $\phi(i)$. The relabelling of the entries in a trade does not change the structure of the corresponding set of edges in $K(r, s, t)$, and we can still decompose these edges into 5-cycles. So for every trade listed here, any relabelling of entries is permissible. In some trades we describe, rows and columns appear adjacent; note that this may not be the case in the latin representation. We define the *transpose* of a trade to be the new trade formed by exchanging rows with columns. Where a particular trade type is referred to, we may sometimes mean the transpose — we do not always distinguish between them.

4.1 Trade type 1

A trade of type 1 always uses twice as many entries from outside the latin rectangle L (in our latin representation M) as inside L. Trades of type 1 involve exchanges of triangles *and* extra edges with 5-cycles; see for example Figure 3. In Table 1 we give three trades of type 1, showing how entries in a latin representation correspond to edges in a complete tripartite graph which in turn may be decomposed into 5-cycles.

4.2 Trade type 2

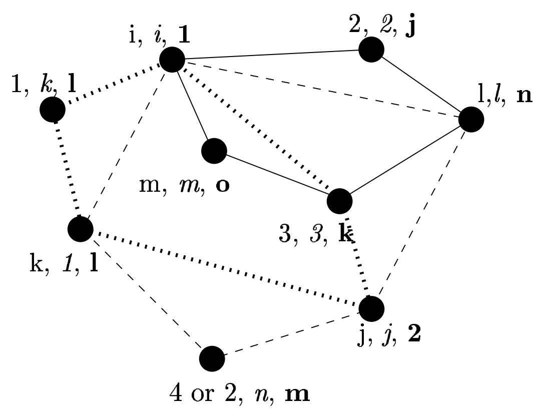
The graphs in Figures 4 and 5 show how five triangles may be exchanged with three 5-cycles to form trades of type 2. The different fonts indicate different labellings of the same graph, from which in turn different trades arise. For example, in Figure 4 a vertex is labelled i, i, **1**. Here i refers to Trades $2A$ and $2B$, i refers to trade $2C$ and **1** refers to Trade $2D$. Because Trades $2A$ and $2B$ differ only slightly we have used the same font. For Trade $2B$ two different vertices are labelled with 2, but not in such a way as to allow multiple edges between vertices.

Note that all trades of type 2 lie entirely within the latin rectangle L, and they do not involve the extra entries in M at the ends of the rows and columns of L.

This completes the list of all the small trades we shall be using. Later we shall

Table 1

Trade type	Entries used in latin representation	Exchange of edges in complete tripartite graphs with 5-cycles
$1A$	$\begin{array}{c\|cc\|\|cc\|} & k & l & & \\ \hline i & 1 & & 2 & 3 \\ j & & 1 & 2 & 3 \\ \hline \end{array}$	$(i_a, k_b, 1_c), (j_a, l_b, 1_c), (i_a, 2_c, j_a, 3_c)$ $\iff (i_a, k_b, 1_c, j_a, 2_c), (i_a, 1_c, l_b, j_a, 3_c)$
$1B$	$\begin{array}{c\|c\|\|cc\|} & k & & \\ \hline i & 1 & 3 & 4 \\ j & 2 & 3 & 4 \\ \hline \end{array}$	$(i_a, k_b, 1_c), (j_a, k_b, 2_c), (i_a, 3_c, j_a, 4_c)$ $\iff (i_a, k_b, 2_c, j_a, 3_c), (i_a, 1_c, k_b, j_a, 4_c)$
$1C$	$\begin{array}{c\|c\|\|cc\|} & m & & \\ \hline i & 1 & & \\ j & 2 & & \\ k & & 1 & 2 \\ l & & 1 & 2 \\ \hline \end{array}$	$(i_a, m_b, 1_c), (j_a, m_b, 2_c), (1_c, k_a, 2_c, l_a)$ $\iff (i_a, m_b, 2_c, l_a, 1_c), (1_c, m_b, j_a, 2_c, k_a)$



	k	l	m
i	1	2	3
j	4	3	

Trade $2A$

	k	l	m
i	1	2	3
j	2	3	

Trade $2B$

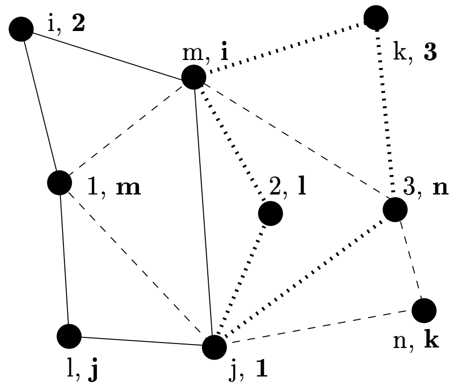
	k	*l*	*m*	*n*
i	*1*	*2*	*3*	
j		*3*		*1*

Trade $2C$

	l	**m**	**n**	**o**
i	**1**	**2**		
j			**1**	
k			**2**	**1**

Trade $2D$

Fig. 4.



	l	m	n
i		1	
j	1	2	3
k		3	

Trade $2E$

	l	**m**	**n**
i	**1**	**2**	**3**
j		**1**	
k			**1**

Trade $2F$

Fig. 5.

introduce some larger trades constructed from these smaller ones.

Now that a variety of trades has been defined, we can use the trades to construct decompositions of complete tripartite graphs into 5-cycles. We saw earlier that each complete tripartite graph has a (not unique) *latin representation*, which consists of a latin rectangle with unordered sets at the end of each row and column. Given a complete tripartite graph (satisfying certain necessary conditions), it is sufficient to find a set of trades from the list mentioned that covers every entry in our latin representation, making sure no two trades overlap. Our task, then, is somewhat like a jigsaw puzzle, where our "pieces" are trades and the "picture" is a latin representation.

We see this process in action in the following example.

Example 12 The graph $K(8, 10, 10)$ admits a decomposition into 5-cycles. Figure 6 gives a suitable latin representation and indicates trades which yield a possible decomposition into 5-cycles. The subscript attached to each entry indicates the trade containing that entry; the key below indicates what type of trade each subscript corresponds to. The numbers in bold always belong to a trade of type 2. To see the trades more clearly in the diagram, the reader may wish to circle entries from different trades with differently coloured pens.

5 All partite sets even

Here we give necessary and sufficient conditions for a decomposition of the complete tripartite graph $K(r, s, t)$ into 5-cycles in the case when r, s and t are all even. Since the case when two partite sets are equal is done in [4], we may assume that $r < s < t$. As some of the explanations in this section are involved, we encourage the reader to refer ahead to Examples 16 and 20 for a better understanding.

1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$\mathbf{2}_{1}$	$\mathbf{1}_{1}$	$\mathbf{4}_{2}$	$\mathbf{3}_{1}$	$\mathbf{6}_{2}$	$\mathbf{5}_{3}$	$\mathbf{8}_{3}$	$\mathbf{7}_{3}$	$\mathbf{10}_{4}$	$\mathbf{9}_{4}$
$\mathbf{3}_{1}$	$\mathbf{4}_{1}$	$\mathbf{5}_{2}$	$\mathbf{6}_{2}$	$\mathbf{7}_{2}$	$\mathbf{8}_{3}$	$\mathbf{9}_{4}$	$\mathbf{10}_{3}$	$\mathbf{1}_{4}$	$\mathbf{2}_{4}$
$\mathbf{4}_{5}$	$\mathbf{3}_{5}$	$\mathbf{6}_{6}$	$\mathbf{5}_{5}$	$\mathbf{8}_{6}$	$\mathbf{7}_{7}$	$\mathbf{10}_{7}$	$\mathbf{9}_{7}$	$\mathbf{2}_{8}$	$\mathbf{1}_{8}$
$\mathbf{5}_{5}$	$\mathbf{6}_{5}$	$\mathbf{7}_{6}$	$\mathbf{8}_{6}$	$\mathbf{9}_{6}$	$\mathbf{10}_{7}$	$\mathbf{1}_{8}$	$\mathbf{2}_{7}$	$\mathbf{3}_{8}$	$\mathbf{4}_{8}$
$\mathbf{6}_{9}$	$\mathbf{5}_{9}$	$\mathbf{8}_{11}$	$\mathbf{7}_{11}$	$\mathbf{10}_{11}$	$\mathbf{9}_{12}$	$\mathbf{2}_{12}$	$\mathbf{1}_{12}$	$\mathbf{4}_{13}$	$\mathbf{3}_{14}$
$\mathbf{7}_{9}$	$\mathbf{8}_{11}$	$\mathbf{9}_{10}$	$\mathbf{10}_{11}$	$\mathbf{1}_{10}$	$\mathbf{2}_{12}$	$\mathbf{3}_{13}$	$\mathbf{4}_{13}$	$\mathbf{5}_{13}$	$\mathbf{6}_{14}$
$\mathbf{8}_{9}$	$\mathbf{7}_{9}$	$\mathbf{10}_{10}$	$\mathbf{9}_{10}$	$\mathbf{2}_{10}$	$\mathbf{1}_{12}$	$\mathbf{4}_{13}$	$\mathbf{3}_{14}$	$\mathbf{6}_{14}$	$\mathbf{5}_{14}$
9	*9*	*1*	*1*	*3*	*3*	*5*	*5*	*7*	*7*
10	*10*	*2*	*2*	*4*	*4*	*6*	*6*	*8*	*8*

Subscripts: $1,2,3,4,5,6,7,8,9,10 : 2A$ $\quad 11 : 2C$ $\quad 12,14 : 2E$ $\quad 13 : 2F$

(Entries in italics are used by trades of type $1B$ in transpose.)

Fig. 6.

The proof of the next lemma follows directly from Corollary 3.

Lemma 13 *If the complete tripartite graph $K(r,s,t)$ (with r, s and t even) decomposes into 5-cycles then one of the following four conditions must hold.*

(*A*) *($r \equiv t \equiv 0$ (modulo 10) and s is not divisible by 10)* or *($s \equiv t$ (modulo 10) and neither s nor t are divisible by 10).*
(*B*) *($s \equiv t \equiv 0$ (modulo 10) and r is not divisible by 10)* or *($r \equiv t$ (modulo 10) and neither r nor t are divisible by 10).*
(*C*) *$r \equiv s$ (modulo 10) and neither r nor s are divisible by 10.*
(*D*) *$r \equiv s \equiv 0$ (modulo 10).*

Lemma 14 *If the complete tripartite graph $K(r,s,t)$ (with r, s and t even) decomposes into 5-cycles then one of the following four conditions must hold.*

1. *$3r - t$ is divisible by 10.*
2. *$3s - t$ is divisible by 10.*
3. *$2r - t$ and $2s - t$ are both divisible by 10.*
4. *r and s are both divisible by 10.*

Moreover, conditions 1, 2, 3 and 4 are implied by conditions (A), (B), (C) and (D) respectively from Lemma 13.

Proof Assume that the complete tripartite graph $K(r,s,t)$ decomposes into 5-cycles. Then one of the conditions (A), (B), (C) or (D) given in the previous lemma is true. Assume that (A) is true. If $r \equiv t \equiv 0$ (modulo 10) then certainly $3r - t$ is divisible by 10. Otherwise $s \equiv t$ (modulo 10) and neither s nor t are divisible by 10. From Theorem 1 we have that $rs + st + rt$ is divisible by 5. Therefore $2rt + t^2 \equiv 0$ (modulo 5), so that $t^2 \equiv 3rt$ (modulo 5) and $t \equiv 3r$ (modulo 5). Since both $3r$ and t are even, $3r - t$ is divisible by 10. Thus if (A) is true condition 1 holds. Similarly it can be shown that if (B) is true condition 2 holds.

Next assume that (C) is true. From Theorem 1 we have that $rs + st + rt$ is divisible by 5. Therefore $2tr + r^2 \equiv 0$ (modulo 5), so that $r^2 \equiv 3tr$ (modulo 5) and $r \equiv 3t$ (modulo 5) or $2r \equiv t$ (modulo 5). Since both $2r$ and t are even, we have that $2r - t$ (and hence $2s - t$) is divisible by 10.

Finally, (D) is equivalent to condition 4. $\square$

The case when r and s are both divisible by 10 is done in [4]. This leaves cases (A), (B) and (C) from Lemma 13, which are dealt with in Subsections 5.1, 5.2 and 5.3 respectively.

In this section we shall need three types of latin representation. We first construct the latin rectangle L_1 and the latin representation M_1, which will be used for most decompositions.

Definition 15 *Let the binary product $i \circ j$ be defined as follows:*

$$i \circ j = \begin{cases} i + j - 3 \ (\textit{modulo } t) & \textit{if } i \textit{ and } j \textit{ are even;} \\ i + j - 1 \ (\textit{modulo } t) & \textit{otherwise.} \end{cases}$$

We define L_1 to be the latin rectangle of dimension $r \times s$ with rows indexed by the set of integers $\{1, 2, \ldots, r\}$ and columns indexed by the set of integers $\{1, 2, \ldots, s\}$ such that the entry $i \circ j$ appears in row i and column j, for $1 \leq i \leq r$ and $1 \leq j \leq s$. Adjoin, to the end of each row and column, entries from the set $\{1, 2, \ldots, t\}$ not already used in that row or column. We define M_1 to be the resultant latin representation.

Example 16 Figure 7 is the latin representation, M_1, of the complete tripartite graph $K(4, 4, 8)$. Note that it contains the latin rectangle L_1.

So consider a latin representation M_1 constructed as in the previous definition. Let P be the set of 2×2 subsquares given by $\{\{(i,j) \mid 2M - 1 \leq i \leq 2M, 2N - 1 \leq j \leq 2N\} \mid 1 \leq M \leq r/2, 1 \leq N \leq s/2\}$. The subsquares of P form a partition of the latin rectangle L_1 contained in M_1. Each time we apply a trade of type $1A$ or $1B$, two cells are used from within the latin rectangle. In our decompositions we shall make sure that these two cells always belong

1	2	3	4	5	6	7	8
2	1	4	3	5	6	7	8
3	4	5	6	7	8	1	2
4	3	6	5	7	8	1	2
5	5	7	7				
6	6	8	8				
7	7	1	1				
8	8	2	2				

Fig. 7.

to the same 2×2 subsquare from P. By doing this we rarely need to worry about what happens to the entries at the ends of the rows and columns; each time we use a trade of type $1A$ we may choose any pair of remaining elements from the particular rows or columns. (This is possible because the entries at the end of each row (or column) $2i-1$ are the same as the entries at the end of each row (or column) $2i$.)

There are $r(t-s)+s(t-r)=t(r+s)-2rs$ entries in our latin representation M_1 not contained in our latin rectangle L_1. Recall that trades of type 1 use half as many entries from inside the latin rectangle as from outside the latin rectangle. So by using $\alpha = (t(r+s)-2rs)/2$ entries from within L_1 with trades of type 1 it is possible to use every entry from outside L_1. This leaves $\beta = (4sr - t(r+s))/2$ entries within L_1. From Theorem 1, it is easy to see that $\beta \geq 0$ and that β is divisible by 10. Therefore it is theoretically possible, at least, to use these β entries with trades of type 2.

Before we begin our decompositions we define some new trades that combine smaller trades of type 1 and 2. Note that, as usual, the transpose of each trade is allowed, entries may be relabelled, and rows and columns rearranged. Trades of type 2 are always in bold, and the subscript numbers are used below each table to identify the trade types.

Note that the trades in Figure 8 occur many times within our latin rectangle L_1. In fact, for $k \geq 3$ any $5 \times k$ or $k \times 5$ subrectangle contained in L_1 (with the columns and rows of the subrectangle adjacent) may be partitioned into thessee trades, which in turn partition into trades of type 2.

However, we may not always be able to partition a 5×2 (or 2×5) subrectangle into trades of type 2. For example, a trade of type $3E$ or $3F$ could be used in rows $2i$ and $2i+1$ but not in rows $2i-1$ and $2i$. We shall make it possible to use 5×2 rectangles in the final two columns. To do this we slightly modify

$\mathbf{1_1}$	$\mathbf{2_1}$	$\mathbf{3_1}$	$\mathbf{4_3}$	$\mathbf{5_2}$
$\mathbf{2_1}$	$\mathbf{1_3}$	$\mathbf{4_3}$	$\mathbf{3_3}$	$\mathbf{6_2}$
$\mathbf{3_1}$	$\mathbf{4_3}$	$\mathbf{5_2}$	$\mathbf{6_2}$	$\mathbf{7_2}$

$1,2:2E \quad 3:2F$

Trade type $3A$

$\mathbf{2_1}$	$\mathbf{1_1}$	$\mathbf{4_3}$	$\mathbf{3_3}$	$\mathbf{6_3}$
$\mathbf{3_1}$	$\mathbf{4_3}$	$\mathbf{5_2}$	$\mathbf{6_3}$	$\mathbf{7_2}$
$\mathbf{4_1}$	$\mathbf{3_1}$	$\mathbf{6_2}$	$\mathbf{5_2}$	$\mathbf{8_2}$

$1,2:2A \quad 3:2C$

Trade type $3B$

$\mathbf{1_1}$	$\mathbf{4_1}$	$\mathbf{3_1}$	$\mathbf{6_2}$	$\mathbf{5_3}$
$\mathbf{4_1}$	$\mathbf{5_2}$	$\mathbf{6_2}$	$\mathbf{7_2}$	$\mathbf{8_3}$
$\mathbf{3_1}$	$\mathbf{6_2}$	$\mathbf{5_3}$	$\mathbf{8_3}$	$\mathbf{7_3}$

$1,3:2E \quad 2:2F$

Trade type $3C$

$\mathbf{2_1}$	$\mathbf{3_3}$	$\mathbf{4_1}$	$\mathbf{5_3}$	$\mathbf{6_2}$
$\mathbf{1_1}$	$\mathbf{4_1}$	$\mathbf{3_3}$	$\mathbf{6_2}$	$\mathbf{5_3}$
$\mathbf{4_1}$	$\mathbf{5_3}$	$\mathbf{6_2}$	$\mathbf{7_2}$	$\mathbf{8_2}$

$1,2:2F \quad 3:2D$

Trade type $3D$

$\mathbf{2_1}$	$\mathbf{1_1}$	$\mathbf{4_2}$	$\mathbf{3_1}$	$\mathbf{6_2}$
$\mathbf{3_1}$	$\mathbf{4_1}$	$\mathbf{5_2}$	$\mathbf{6_2}$	$\mathbf{7_2}$

$1,2:2A$

Trade type $3E$

$\mathbf{1_1}$	$\mathbf{4_1}$	$\mathbf{3_1}$	$\mathbf{6_2}$	$\mathbf{5_2}$
$\mathbf{4_1}$	$\mathbf{5_2}$	$\mathbf{6_1}$	$\mathbf{7_2}$	$\mathbf{8_2}$

$1,2:2A$

Trade type $3F$

$\mathbf{1_2}$	$\mathbf{2_2}$	$\mathbf{3_3}$	$\mathbf{4_2}$	$\mathbf{5_3}$
$\mathbf{2_1}$	$\mathbf{1_1}$	$\mathbf{4_1}$	$\mathbf{3_1}$	$\mathbf{6_1}$
$\mathbf{3_1}$	$\mathbf{4_1}$	$\mathbf{5_1}$	$\mathbf{6_1}$	$\mathbf{7_1}$
$\mathbf{4_2}$	$\mathbf{3_2}$	$\mathbf{6_3}$	$\mathbf{5_3}$	$\mathbf{8_3}$

$1:3E \quad 2,3:2A$

Trade type $3G$

$\mathbf{2_1}$	$\mathbf{3_1}$	$\mathbf{4_1}$	$\mathbf{5_4}$	$\mathbf{6_4}$
$\mathbf{1_1}$	$\mathbf{4_1}$	$\mathbf{3_3}$	$\mathbf{6_3}$	$\mathbf{5_3}$
$\mathbf{4_2}$	$\mathbf{5_2}$	$\mathbf{6_3}$	$\mathbf{7_4}$	$\mathbf{8_3}$
$\mathbf{3_2}$	$\mathbf{6_2}$	$\mathbf{5_2}$	$\mathbf{8_4}$	$\mathbf{7_4}$

$1,2,3,4:2A$

Trade type $3H$

Fig. 8.

the structure of our latin representation.

Definition 17 *Let R be a set of consecutive numbers from the set $\{1, 2, \ldots, r\}$ such that $|R|$ is divisible by* 5*, and let $t - s \geq 4$. Take the latin rectangle L_1 (of order $r \times s$) constructed for M_1. Consider the intersection of the first* 5 *rows labelled by R with the final two columns $s-1$ and s, as shown in Figure* 9*. (The actual entries will be some relabelling of the entries in this figure.)*

Now swap two entries, as shown in Figure 10*.*

Repeat this procedure for each set of five consecutive row numbers in R. To avoid repeated elements in the rows indexed by R, add 2 *(modulo t) to each*

$s-1$	s
1	2
2	1
3	4
4	3
5	6

(first row number in R is odd)

or

$s-1$	s
6	5
7	8
8	7
9	10
10	9

(first row number in R is even)

Fig. 9.

	$s-1$	s
	1	2
$\rightarrow$	*3*	1
$\rightarrow$	*2*	4
	4	3
	5	6

(first row number in R is odd)

or

	$s-1$	s
	6	5
	7	8
$\rightarrow$	*9*	7
$\rightarrow$	*8*	10
	10	9

(first row number in R is even)

Fig. 10.

entry in the final two columns. (The condition $t-s \geq 4$ ensures no entries are repeated in the rows.) Finally add entries from the set $\{1, 2, \ldots, t\}$ not already used in each row or column to the end of that row or column. We define $M_2(R)$ to be the resultant latin representation.

Since $M_2(R)$ is very similar to M_1, most of our observations on M_1 still hold. We may apply the same partition P of our latin rectangle, ensuring that each trade of type 1 uses two entries from the same 2×2 subsquare. However, we must take care when selecting entries from the ends of the rows. We no longer always have the entries at the end of row $2i$ being the same as the entries at the end of row $2i-1$.

Any 5×2 sub-rectangle (within the rows labelled by R) of the final 2 columns may be taken up by trades of type 2, as shown in Figure 11. To apply these trades we require that $t - s \geq 6$.

The entries subscripted with 3 and 4 are *not* part of trades $3I$ and $3J$. Rather, *we must use these entries with trades of type $1B$ at some point to ensure that the remaining entries at the ends of the rows labelled by R may be paired off as usual.* Columns $2c-1$ and $2c$ are adjacent columns with $c < s$; we specify

$2c-1$	$2c$	$\dots$	$s-1$	s	
x_3		$\dots$	$\mathbf{1_1}$	$\mathbf{2_1}$	$3_3\ 5_3$
	x_4	$\dots$	$\mathbf{3_1}$	$\mathbf{1_1}$	$2_4\ 5_4$
y_3		$\dots$	$\mathbf{2_1}$	$\mathbf{4_2}$	$3_3\ 5_3$
	y_4	$\dots$	$\mathbf{4_2}$	$\mathbf{3_2}$	$2_4\ 5_4$
			$\mathbf{5_2}$	$\mathbf{6_2}$	

$1 : 2B \quad 2 : 2A$

Trade type $3I$

(first row number is odd)

$2c-1$	$2c$	$\dots$	$s-1$	s	
			$\mathbf{6_2}$	$\mathbf{5_2}$	
x_3		$\dots$	$\mathbf{7_1}$	$\mathbf{8_1}$	$9_3\ 11_3$
	x_4	$\dots$	$\mathbf{9_1}$	$\mathbf{7_1}$	$8_4\ 11_4$
y_3		$\dots$	$\mathbf{8_1}$	$\mathbf{10_2}$	$9_3\ 11_3$
	y_4	$\dots$	$\mathbf{10_2}$	$\mathbf{9_2}$	$8_4\ 11_4$

$1 : 2B \quad 2 : 2A$

Trade type $3J$

(first row number is even)

Fig. 11.

$\mathbf{1_1}$	$\mathbf{2_1}$	3_5	4_5	$\mathbf{5_2}$	$\mathbf{6_2}$	7_6	8_6	$\mathbf{9_1}$	$\mathbf{10_2}$
$\mathbf{2_1}$	1_3	4_4	3_4	$\mathbf{6_2}$	5_3	8_7	7_7	$\mathbf{10_1}$	$\mathbf{9_2}$
3_5	3_3	1_4	1_4	3_5	3_3	5_7	5_7	7_6	7_6
4_5	4_3	2_4	2_4	4_5	4_3	6_7	6_7	8_6	8_6

$1,2 : 2A \quad 3,4,7 : 1B \quad 5,6 : 1C$

Fig. 12. Trade type $3K$

these columns when the trades are applied.

We can also construct a latin representation $M_3(C)$, where C is a set of consecutive column numbers and $|C|$ is divisible by 5. In this case take the latin rectangle of order $r \times s$ constructed for M_1, and swap some entries in the columns labelled by C in the same way as before, except in transpose. We then add 2 (modulo t) to every entry in the final *rows* $r-1$ and r. Finally we add entries from the set $\{1, 2, \dots, t\}$ not used in each row and column to the end of that row or column.

The trades $3I$ and $3J$ may now be applied in transpose, but only if $t - r \geq 6$. The trade in Figure 12 also can only be used in a latin representation of type $M_3(C)$ (or $M_2(R)$ if used in transpose).

Trades of type $3I$, $3J$ and $3K$ all specify entries from outside the latin rectangle. Therefore they should be applied first to ensure that the entries outside the rectangle have not already been used. For convenience and clarity, however, in our constructions they may not be mentioned first.

In this section, trades of type 1 that use entries from the ends of the rows will be suffixed by (ac) (meaning "across") while trades of type 1 that use entries from the bottoms of the columns will be suffixed by (dwn) (meaning "down"). (This notation will only apply to trades of type $1A$ and $1B$. Trades of type $1D$ are used only in trades of type $3K$.) So if $(t-r)/2$ entries in each row are used by trades of type 1(dwn) and $(t-s)/2$ entries in each column are used by trades of type 1(ac), we have used all entries from outside the latin rectangle L.

5.1 $3r - t$ divisible by 10

This subsection deals with Case (A) from Lemma 13.

The following lemma will be used in Theorem 19. It will confirm that in the situations where we wish to use trades of type $3I$, $3J$ or $3K$, we have $t-s \geq 6$ as required.

Lemma 18 *Assume that $r < s < t$, $3r - t$ is divisible by 10, and let*

$$f = \lfloor \frac{(t-s)r/2}{(3r-t)} \rfloor.$$

Let $K(r,s,t)$ be a complete tripartite graph satisfying the necessary conditions of Theorem 1. If $s - 2f \leq 4$, then $t - s \geq 6$.

Proof Assume the converse, that is, $s-2f \leq 4$ and $t-s \leq 4$. From Corollary 2 we have $t \leq 3r$. If $t = 3r$, from condition (iii) of Theorem 1 it follows that $s = t$, which contradicts our assumption that $s < t$. But since $3r-t$ is divisible by 10, we must have $3r - t \geq 10$.

Therefore $4 \geq s - 2f \geq s - (t-s)r/(3r-t) \geq s - 4r/10$. Equivalently, $2r \geq 5(s-4)$. From Corollary 3, at least two partite sets are congruent modulo 5. Since $r < s < t$ and $t - s \leq 4$, we must have $s - r \geq 6$. Therefore $2r \geq 5(s-4) \geq 5(r+2)$, a contradiction. $\square$

Theorem 19 *Let $K(r,s,t)$ be a complete tripartite graph with $r < s < t$ satisfying the necessary conditions given in Theorem 1, and such that either ($r \equiv t \equiv 0$ (modulo 10) and s is not divisible by 10) or ($s \equiv t$ (modulo 10) and neither s nor t is divisible by 10). Then $K(r,s,t)$ may be decomposed into 5-cycles.*

Proof From Lemma 14, $3r - t$ is divisible by 10.

First the general idea behind the proof of this theorem will be discussed. Consider the latin representation M_1, constructed in Definition 15. Let the latin rectangle inside our latin representation be L. There are $t-r$ entries at the bottom of each column. Recall that trades of type 1 use twice as many entries from outside L as from inside L. Therefore in each column we must use $(t-r)/2$ entries with trades of type 1(dwn), using all the entries from the bottom of the columns. This leaves $r-(t-r)/2=(3r-t)/2$ entries in each column, to be used by trades of type 1(ac) or trades of type 2. There are $(t-s)r$ entries at the ends of the rows, which must also be used by trades of type 1. For each *pair* of columns (beginning from the left) we use $(t-r)$ entries with trades of type 1(dwn) and $3r-t$ entries with trades of type 1(ac), until all the entries from the ends of the rows are used. This requires the use of at least $(t-s)r/(2(3r-t))$ pairs of columns.

So let

$$(t-s)r/2=(3r-t)f+g, \tag{1}$$

where f and g are integers such that $f \geq 0$ and $0 \leq g < (3r-t)$. Note that this f is the same as that used in Lemma 18. From Theorem 1 we know that $t \leq 4rs/(r+s)$. Equivalently, $(t-s)r/2 \leq (3r-t)s/2$, which implies that f is less than or equal to $s/2$.

In each of the first f pairs of columns, we use $t-r$ entries with trades of type 1(dwn), spreading these entries as evenly as possible along the columns (see Figure 13). The remaining $3r-t$ entries from each of these pairs of columns as well as g entries from columns $2f+1$ and $2f+2$ are used by trades of type 1(ac), using all entries from the ends of the rows.

From Lemma 14 we know that $3r-t$ is divisible by 10. Since either $t-s$ or r is divisible by 10 we must have that g is divisible by 10. In the final $s-2f$ columns, trades of type 2 are used. In each column (except possibly columns $2f+1$ and $2f+2$) entries in the final $(t-r)/2$ rows are used by trades of type 1(dwn). There are $(3r-t)/2$ entries left in each column, but since this number is divisible by 5 what remains is a rectangle of entries with a side divisible by 5. In most cases, then, it is easy to use this rectangle with trades of type $3A$ to $3H$.

This is the idea behind the decompositions in this theorem. Each case in the proof rests upon the fact that $3r-t$ is divisible by 10, which makes the trades of type 2 easy to place. Care must be taken in columns $2f+1$ and $2f+2$, where trades of type 1(ac), 1(dwn) *and* trades of type 2 may be used. Other special cases arise when $s-2f$ is small, and the latin representation $M_2(R)$ is needed.

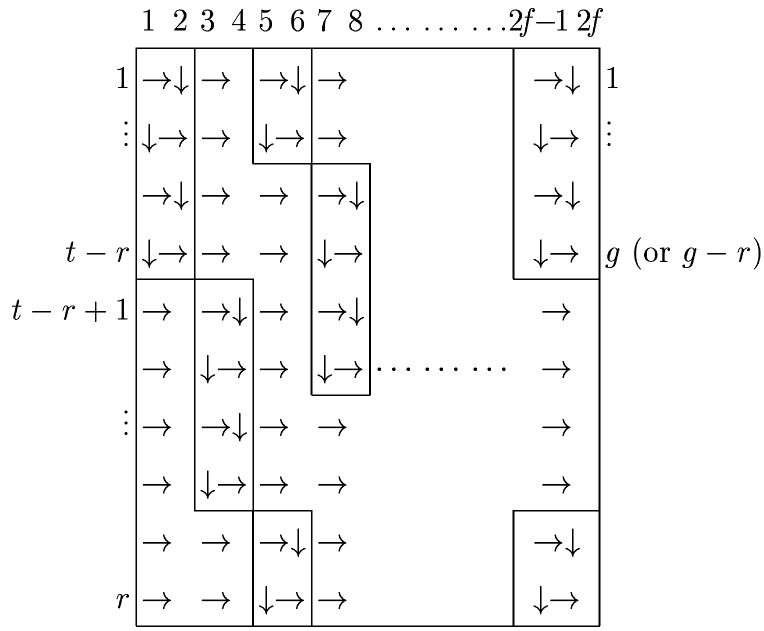


Fig. 13. Columns 1 to $2f$ of our latin rectangle.

<u>Columns 1 to $2f$</u>

We start by concentrating on the entries in the first $2f$ columns of L. Our latin representation at this point can be either M_1 or $M_2(R)$ (for some set of consecutive row labels R). The latin representation $M_3(C)$ is *not* used in this theorem.

First consider when $t < 2r$.

The following set of cells containing *even* entries is used by trades of type $1A$(dwn), using all $t-r$ entries from the ends of columns 1 to $2f$: $\{(2i, 2j-1), (2i-1, 2j) \mid 1 \leq j \leq f, \quad (j-1)(t-r)/2+1 \leq i \leq j(t-r)/2\}$, where $2i$ and $2i-1$ are calculated modulo r. All remaining entries in the first $2f$ columns are used by trades of type $1A$(ac). (Some of the trades of type $1A$(ac) may later be replaced with trades of type $1B$(ac) when using the latin representation $M_2(R)$.)

This process can be seen more clearly in Figure 13. (The arrows in this and other diagrams in the chapter indicate whether trades of type 1(ac) or 1(dwn) are being used.)

In each pair of columns, $t-r$ rows are used by trades of type 1(ac) and 1(dwn) while all the remaining $2r-t$ rows are used by trades of type 1(ac) only, as indicated by the arrows. Note that trades of type 1(ac) have been spread as evenly as possible across the rows. We have used $2(3r-t)f$ entries from the ends of the rows, so there are $2g$ entries remaining. If $g < r$ there are two entries remaining at the ends of rows 1 to g; otherwise $g \geq r$ and there are

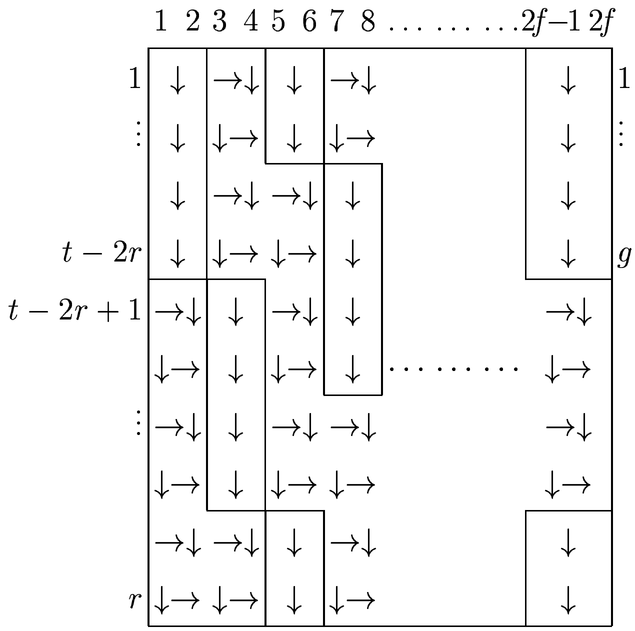


Fig. 14. Columns 1 to $2f$ of our latin rectangle.

four entries remaining at the ends of rows 1 to $g-r$ and two entries at the ends of the remaining rows.

Next consider when $t \geq 2r$.

The following set of cells containing *odd* entries is used by trades of type $1A(\text{dwn})$, using $t-2r$ entries from the ends of columns 1 to $2f$: $\{(2i-1, 2j-1), (2i, 2j) \mid 1 \leq j \leq f, \quad (j-1)(t-2r)/2+1 \leq i \leq j(t-2r)/2\}$, where $2i$ and $2i-1$ are calculated modulo r. All even entries in the first $2f$ columns are also used by trades of type $1A(\text{dwn})$. Remaining odd entries are used by trades of type $1A(\text{ac})$. (As in the case when $t < 2r$, some of the trades of type $1A(\text{ac})$ may later be replaced with trades of type $1B(\text{ac})$ when using the latin representation $M_2(R)$.)

This process can be seen more clearly in Figure 14. In each pair of columns, $t-2r$ rows are used by trades of type $1(\text{dwn})$ only. The remaining $3r-t$ rows are used by trades of type $1(\text{dwn})$ and $1(\text{ac})$. This is indicated by the arrows in Figure 14. Note that trades of type $1(\text{ac})$ have been spread as evenly as possible across the rows. We have used $2(3r-t)f$ entries from the ends of the rows, so there are $2g$ entries remaining. Since $t \leq 2r$ and $g < 3r-t$, we have $g < r$. Therefore there are two remaining entries at the ends of rows 1 to g.

If $s = 2f$ our decomposition is complete. Otherwise assume that $s > 2f$. Now we describe trades to use the entries of the final $s-2f$ columns of L. There are three cases to consider.

<u>Case 1: $g \geq r$.</u> Since r and s are not equal, we may say that $r \leq s-2$, or

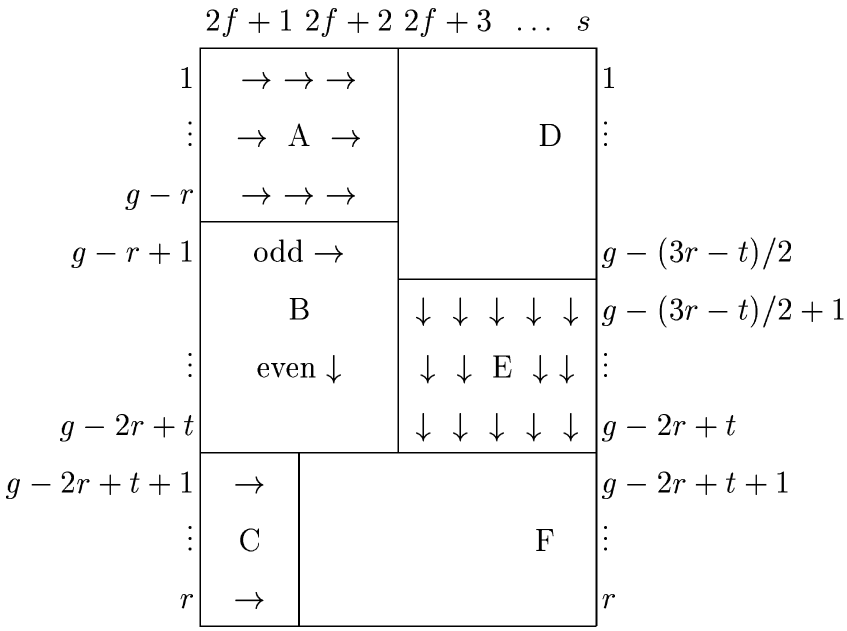


Fig. 15.

$2r \leq 2s - 4$. Since $g \geq r$ we must have $t < 2r$, and it follows that $t - s \leq 2r - s - 2 \leq s - 6$. From equation (1) and the fact that $3r - t > r$ we have $f < (t - s)/2 \leq s/2 - 3$. Thus f is at most $s/2 - 4$, and after the first $2f$ columns have been used we have at least eight columns left. This means that $s - 2f$ is large enough for us to use the latin representation M_1.

The entries of columns $2f + 1$ to s are used as shown in Figure 15.

Entries in A and the odd entries in B are used by trades of type $1A$(ac), using all remaining entries from the ends of these rows. The even entries in B are used by trades of type $1A$(dwn), using all entries from the bottom of columns $2f+1$ and $2f+2$. The entries of C are used by trades of type $1B$(ac), while the entries of E are used by trades of type $1B$(dwn). Thus all entries from outside the latin rectangle have been used. Since g and $3r - t$ are both divisible by 10, the number of rows in D and F are divisible by 5. Also $s - (2f + 3) \geq 5$, and so D and F may be taken up by trades of type $3A$ to $3H$.

<u>Case 2: $g < r$, $g \leq (3r - t)/2$.</u>

We use the final $s - 2f$ columns as shown in Figure 16.

If $s - 2f = 2$, we use the latin representation $M_2(R)$, where $R = \{g + 1, \ldots, (3r - t)/2\}$. This is possible because g and $3r - t$ are both divisible by 10. Also, from Lemma 18 we have that if $s - 2f \leq 4$ then $t - s \geq 6$. In this case A and B are used by trades of type $3K$. The entries of C are used by trades of type $3I$ and $3J$. For these trades we may set $c = f$ (see Definition 17 and the following comments). Thus some entries in columns $2f - 1$ and

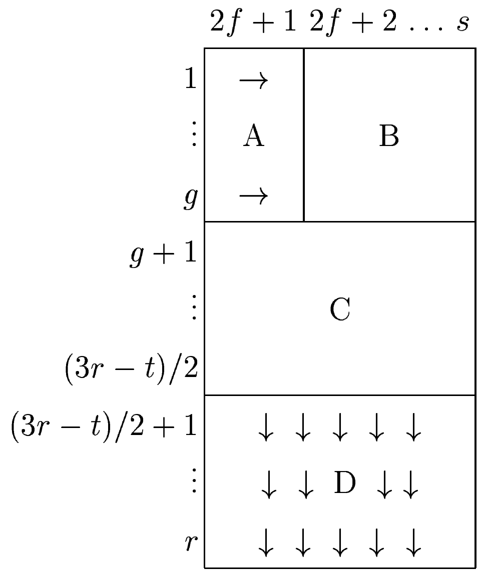


Fig. 16.

$2f$ are used by trades of type $1B(\text{ac})$ rather than $1A(\text{ac})$. This is fine, since Figures 13 and 14 confirm that the odd entries in columns $2f-1$ and $2f$ of rows $g+1$ to $(3r-t)/2$ are used by trades of type $1(\text{ac})$.

Otherwise $s - 2f \geq 4$ and we use the latin representation M_1. The entries of A are used by trades of type $1B(\text{ac})$, using all remaining entries from the ends of the rows. The entries of B and C are used by trades of type $3A$ to $3H$. The entries of D are taken up by trades of type $1B(\text{dwn})$, using all remaining entries from the bottoms of the columns.

Finally we have the case when g is less than r but greater than $(3r-t)/2$.

Case 3: $r > g > (3r-t)/2$. The final $s - 2f$ columns are used as shown in Figure 17.

The odd entries in C are used by trades of type $1A(\text{ac})$ while the even entries in C are used by trades of type $1A(\text{dwn})$, using all entries from the bottom of columns $2f+1$ and $2f+2$.

If $s - 2f = 2$, we use the latin representation $M_2(R)$, where $R = \emptyset$. In this case D and F do not exist. Entries in A and B are used by trades of type $3K$. The entries of E are used by trades of type $1B(\text{dwn})$.

Next consider when $s - 2f = 4$. Here we use the latin representation $M_2(R)$, where $R = \{3r-t-g+1, \dots, (3r-t)/2\}$. The entries of D are used by trades of type $3I$ and $3J$, setting $c = f+1$ (see Definition 17 and the comments that follow). Note that the odd entries of C (in columns $2f+1$ and $2f+2$) are used by trades of type $1(\text{ac})$, as needed. Entries in A are used by trades of

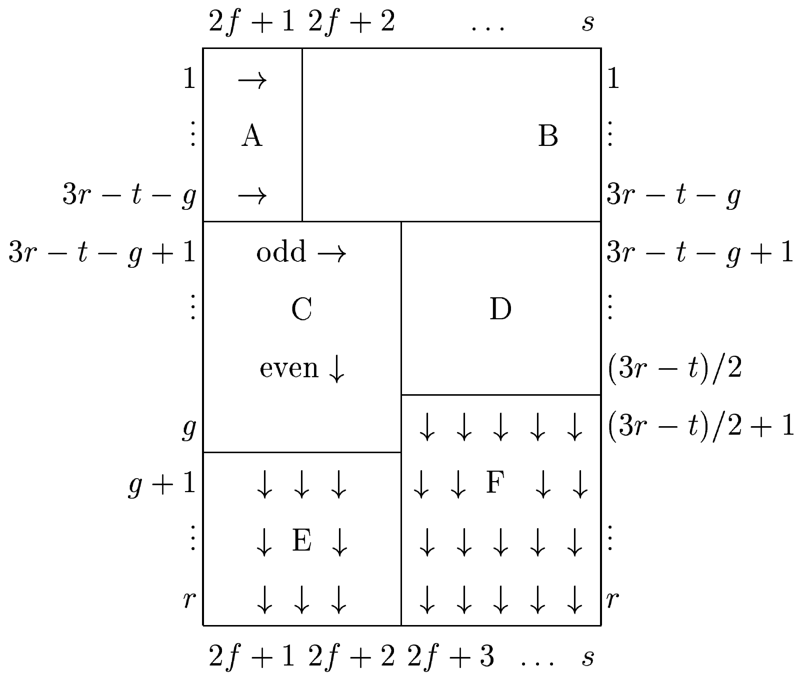


Fig. 17.

type $1A$(ac). The entries of E and F are used by trades of type $1B$(dwn). The entries of B are used by trades of type $3A$ to $3H$.

Finally, consider when $s - 2f \geq 6$. Here we use the latin representation M_1. Entries in A are taken up by trades of type $1A$(ac). The entries of E and F are used by trades of type $1B$(dwn). The subarrays B and D have $(3r-t)-g$ and $g-(3r-t)/2$ rows respectively. Since g and $3r-t$ are both divisible by 10, The entries of B and D may be used by trades of type $3A$ to $3H$.

This completes all cases. □

Example 20 We show how the complete tripartite graph $K(12, 16, 26)$ may be decomposed into 5-cycles. Our latin rectangle L is as shown in Figure 18.

Note that $(t-s)r/2 = 60$ and $3r - t = 10$. Therefore $f = 6$ and $g = 0$, so we have $t \geq 2r$, $s \geq 2f + 4$ and $g < (3r-t)/2$, an example of Case 2 in the previous theorem. Since $g = 0$, A and B are non-existent and only the entries of C and D occur. Entries in bold are used by a trade of type $3J$, entries in italics are used by trades of type $1A$(ac) and remaining entries are used by trades of type $1A$(dwn).

1	2	3	*4*	5	*6*	7	*8*	9	*10*	11	*12*	**13**	**14**	**15**	**16**
2	1	*4*	3	*6*	5	*8*	7	*10*	9	*12*	11	**14**	**13**	**16**	**15**
3	*4*	5	6	7	*8*	9	*10*	11	*12*	13	*14*	**15**	**16**	**17**	**18**
4	3	6	5	*8*	7	*10*	9	*12*	11	*14*	13	**16**	**15**	**18**	**17**
5	*6*	7	*8*	9	10	11	*12*	13	*14*	15	*16*	**17**	**18**	**19**	**20**
6	5	*8*	7	10	9	*12*	11	*14*	13	*16*	15	18	17	20	19
7	*8*	9	*10*	11	*12*	13	14	15	*16*	17	*18*	19	20	21	22
8	7	*10*	9	*12*	11	14	13	*16*	15	*18*	17	20	19	22	21
9	*10*	11	*12*	13	*14*	15	*16*	17	18	19	*20*	21	22	23	24
10	9	*12*	11	*14*	13	*16*	15	18	17	*20*	19	22	21	24	23
11	*12*	13	*14*	15	*16*	17	*18*	19	*20*	21	22	23	24	25	26
12	11	*14*	13	*16*	15	*18*	17	*20*	19	22	21	24	23	26	25

Fig. 18.

5.2 $3s-t$ *divisible by* 10

This subsection deals with Case (B) from Lemma 13. We first require the following lemma. It confirms that $t-r \geq 6$ for construction of the latin representation $M_3(C)$.

Lemma 21 *Assume that $r<s<t$ and that the graph $K(r,s,t)$ satisfies the necessary conditions of Theorem* 1*. Then $t-r \geq 6$.*

Proof From Corollary 3, at least two partite sets are congruent modulo 10. Since $r<s<t$, it follows that $t-r \geq 10$. So certainly $t-r \geq 6$. □

Theorem 22 *Let $K(r,s,t)$ be a complete tripartite graph with $r<s<t$ satisfying the necessary conditions given in Theorem* 1 *such that either ($s \equiv t \equiv 0$ (modulo* 10*) and r is not divisible by* 10*)* or *($t \equiv r$ (modulo* 10*) and neither t nor r is divisible by* 10*). Then $K(r,s,t)$ may be decomposed into* 5*-cycles.*

Proof The following proof is quite similar to that of the previous theorem, except in transpose.

From Lemma 14 we know that $3s-t$ is divisible by 10. Let

$$(t-r)s/2=(3s-t)f+g,$$

where f and g are integers such that $f \geq 0$ and $0 \leq g < (3s-t)$. Since either $t-r$ or s is divisible by 10 we must have g divisible by 10. Also, since $t \leq 2s$ (see Corollary 2), g is less than s. From Theorem 1 we may assume that $t \leq 4rs/(r+s)$. Equivalently, $(t-r)s/2 \leq (3s-t)r/2$, which implies that f is less than or equal to $r/2$.

Rows 1 to $2f$

We use the entries in rows $1, 2, \ldots, 2f$ as follows. At this stage it is unspecified whether the latin representation M_1 or $M_3(C)$ is being used. The latin representation $M_2(R)$ is not seen in this theorem.

The following set of cells with even entries is used by trades of type $1A$(ac), using all $t-s$ entries from the ends of rows 1 to $2f$: $\{(2i, 2j-1), (2i-1, 2j) \mid 1 \leq i \leq f, \quad (i-1)(t-s)/2+1 \leq j \leq i(t-s)/2\}$, where $2j$ and $2j-1$ are calculated modulo s. All remaining entries in the first $2f$ rows are used by trades of type $1A$(dwn). (Some of the trades of type $1A$(dwn) may later be replaced with trades of type $1B$(dwn) when using the latin representation $M_2(C)$.)

This process can be seen more clearly in Figure 19. 1 to $2f$.

In each pair of rows, the even entries in $t-s$ columns are used by trades of type 1(ac), using all the entries from the ends of the first $2f$ rows. In the remaining $2s-t$ columns of each pair of rows, half of the entries are used by trades of type 1(ac) and half by trades of type 1(dwn).

If $r = 2f$ we are finished. Otherwise assume that $r > 2f$.

Rows $2f+1$ to r.

Since $g < r$ there are only two cases, similar to Cases 2 and 3 in Theorem 19.

Case 1: $g < (3s-t)/2$. The final $r-2f$ rows are used as shown in Figure 20.

The entries of D are used by trades of type $1A$(ac).

If $r = 2f+2$, we use the latin representation $M_2(\{g+1, \ldots, (3s-t)/2\})$, where $\{g+1, \ldots, (3s-t)/2\}$ is a set of columns. The entries of A and B are used by trades of type $3K$. The entries of C are taken up by trades of type $3I$ and $3J$, with $c = f$ (see Definition 17 and the comments that follow).

Otherwise $r \geq 2f+4$, and we use the latin representation M_1. The entries of A are used by trades of type $1A$(dwn), while the entries of B and C are used by trades of type $3A$ to $3H$.

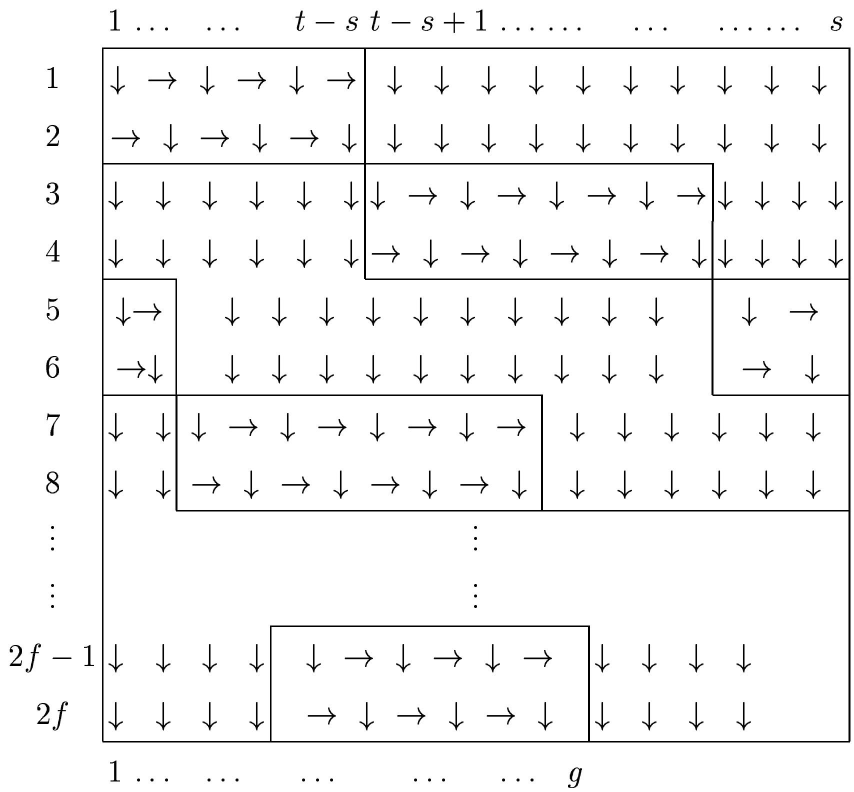


Fig. 19. Rows 1 to $2f$.

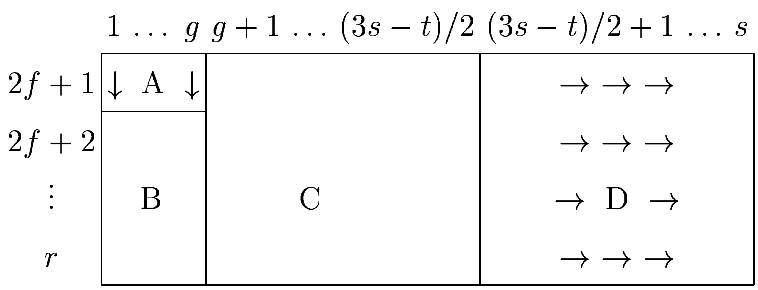


Fig. 20.

Case 2: $g \geq (3s-t)/2$. The final $r-2f$ rows are used as shown in Figure 21.

All entries of E and the even entries of C are used by trades of type $1A$(ac). The odd entries of C are used by trades of type $1A$(dwn).

If $r = 2f+2$, we require the latin representation $M_3(C)$, with $C = \emptyset$. D and F do not exist. We use A and B with trades of type $3K$.

If $r = 2f+4$, the latin representation $M_3(C)$ is needed, where $C = \{3s-t-g+1, \ldots, (3s-t)/2\}$. The entries of A are used by trades of type $1A$(dwn), while the entries of B are used by trades of type $3B$ and $3C$. The entries of D are used by trades of type $3I$ and $3J$, with $c = f+1$. The entries of F are

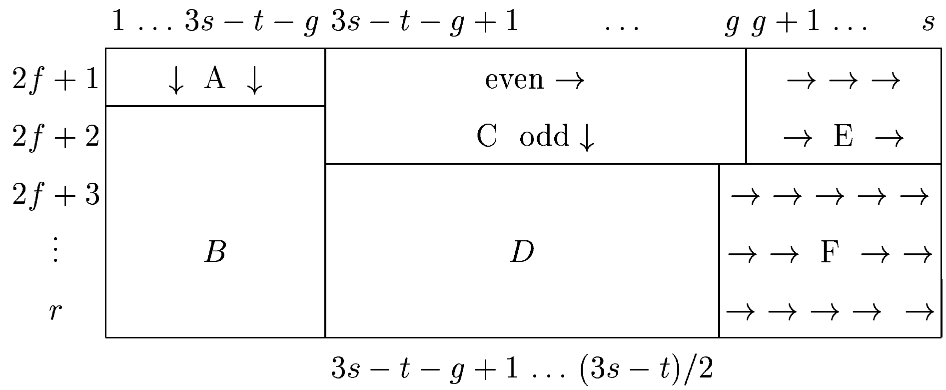


Fig. 21.

taken up by trades of type $1A$(ac).

Otherwise $r \geq 2f+6$, and M_1 is used. The entries of A are taken up by trades of type $1A$(dwn), while B and D are used by trades of type $3A$ to $3I$. The entries of F are taken up by trades of type $1A$(ac).

This completes every case. □

For the next subsection, we construct some new trades, combining smaller trades of type 2.

The trades in Figure 22 use only half of the entries within a subrectangle. As shown, these trades use all odd entries in a subrectangle; however, by a relabelling of entries and swapping rows and columns, the trades may use all even entries of a subrectangle instead.

5.3 $2r-t$ *and* $2s-t$ *divisible by* 10

This subsection deals with Case (C) given in Lemma 13.

Theorem 23 *Let $K(r,s,t)$ be a complete tripartite graph with $r<s<t$ satisfying the necessary conditions given in Theorem* 1 *such that $r \equiv s$ (modulo* 10*) and neither r nor s is divisible by 10. Then $K(r,s,t)$ may be decomposed into* 5*-cycles.*

Proof From Lemma 14 we see that both $2r-t$ and $2s-t$ are divisible by 10. We consider two cases, $t<2r$ and $t \geq 2r$.

1_1		3_1		5_1		7_2		9_2	
	1_3		3_3		5_3		7_4		9_4
3_1		5_1		7_2		9_2		11_2	
	3_3		5_3		7_4		9_4		11_4

$1, 2, 3, 4 : 2B$

Trade type $3L$

1_1		3_1		5_1		7_3		9_3	
	1_4		3_4		5_4		7_6		9_6
3_1		5_2		7_2		9_2		11_3	
	3_4		5_5		7_5		9_5		11_6
5_1		7_2		9_2		11_3		13_3	
	5_4		7_5		9_5		11_6		13_6

$1, 4 : 2E \quad 2, 5 : 2B \quad 3, 6 : 2A$

Fig. 22. Trade type $3M$

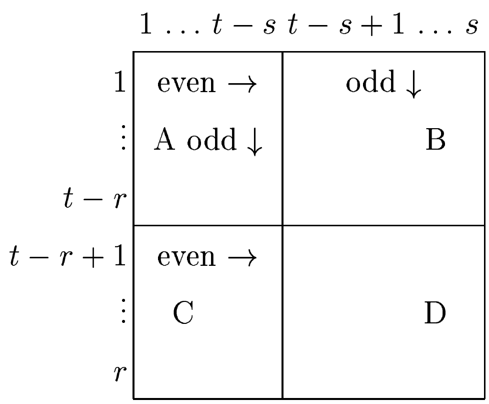


Fig. 23.

<u>Case 1: $t < 2r$.</u> Our latin representation for this case is M_1. We partition our latin rectangle into four submatrices as shown in Figure 23.

The even entries of A and C are used by trades of type $1A$(ac), while the odd entries of A and B are used by trades of type $1A$(dwn). This uses all entries from outside the latin rectangle. Since $2r - t$ is divisible by 10, the odd entries of C and D may be used by trades of type $3L$ and $3M$. Similarly since $2s - t$ is divisible by 10, the even entries of the final $2s - t$ columns may be used.

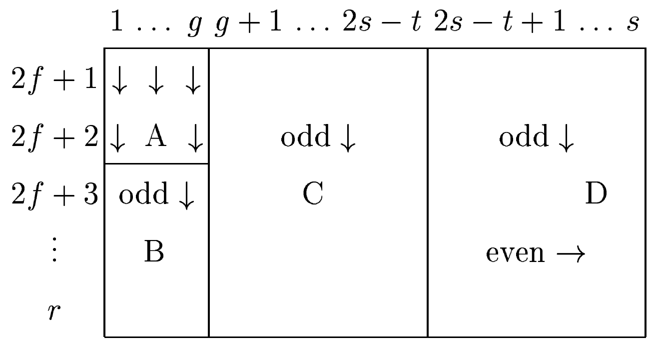


Fig. 24.

<u>Case 2: $t \geq 2r$.</u> The idea behind this case is to use half the entries in our latin rectangle with trades of type $1A$(dwn). This leaves $t-2r$ entries at the bottom of each column, so we need to use $s(t-2r)/2$ more entries from within our latin rectangle with trades of type 1(dwn).

So let $s(t-2r)/2 = (2s-t)f + g$ where f and g are non-negative integers such that $g < (2s-t)$. From Theorem 1 we may assume that $t(r+s) \leq 4rs$, or equivalently $(t-2r)s/2 \leq r(2s-t)/2$, so f is no greater than $r/2$.

We first use the entries in rows 1 to $2f$. The following set of cells containing even entries is used by trades of type $1A$(ac), using all $t-s$ entries from the ends of rows 1 to $2f$: $\{(2i, 2j-1), (2i-1, 2j) \mid 1 \leq i \leq f, \quad (i-1)(t-s)/2+1 \leq j \leq i(t-s)/2\}$, where $2j$ and $2j-1$ are calculated modulo s. All remaining entries in the first $2f$ rows are used by trades of type $1A$(dwn).

If $r = 2f$ our decomposition is complete. Otherwise we use the final $r-2f$ rows as shown in Figure 24.

The entries of A and the odd entries of D are used by trades of type $1A$(dwn). The even entries of D are used by trades of type $1A$(ac), using all entries from the ends of the final $r-2f$ rows.

If $r = 2f+2$, we use a latin representation of type $M_2(R)$, where $R = \emptyset$. The entries of C are used by trades of type $3K$. B does not exist in this case.

If $r = 2f+4$, a latin representation of type $M_2(\emptyset)$ is again needed. The entries of B are used by trades of type $3K$. The odd entries of C are used by trades of type $1A$(dwn), while the even entries are used by trades of type $3L$.

Finally if $r \geq 2f+6$, M_1 is used. The odd entries of B and C are taken up by trades of type $1A$(dwn), while the even entries of B and C are used by trades of type $3L$ and $3M$.

We have now used every entry in our latin representation, so the decomposition is complete. $\square$

1	*2*	3	$\mathbf{4_1}$	5	$\mathbf{6_1}$	7	$\mathbf{8_1}$	9	$\mathbf{10_1}$	11	$\mathbf{12_1}$
2	1	$\mathbf{4_1}$	3	$\mathbf{6_1}$	5	$\mathbf{8_1}$	7	$\mathbf{10_1}$	9	$\mathbf{12_1}$	11
$\mathbf{3_3}$	*4*	$\mathbf{5_3}$	$\mathbf{6_1}$	$\mathbf{7_3}$	$\mathbf{8_1}$	$\mathbf{9_4}$	$\mathbf{10_1}$	$\mathbf{11_4}$	$\mathbf{12_1}$	$\mathbf{13_4}$	$\mathbf{14_1}$
4	$\mathbf{3_3}$	$\mathbf{6_1}$	$\mathbf{5_3}$	$\mathbf{8_1}$	$\mathbf{7_3}$	$\mathbf{10_1}$	$\mathbf{9_4}$	$\mathbf{12_1}$	$\mathbf{11_4}$	$\mathbf{14_1}$	$\mathbf{13_4}$
$\mathbf{5_3}$	*6*	$\mathbf{7_3}$	$\mathbf{8_1}$	$\mathbf{9_3}$	$\mathbf{10_1}$	$\mathbf{11_4}$	$\mathbf{12_1}$	$\mathbf{13_4}$	$\mathbf{14_1}$	$\mathbf{1_4}$	$\mathbf{2_1}$
6	$\mathbf{5_3}$	$\mathbf{8_1}$	$\mathbf{7_3}$	$\mathbf{10_1}$	$\mathbf{9_3}$	$\mathbf{12_1}$	$\mathbf{11_4}$	$\mathbf{14_1}$	$\mathbf{13_4}$	$\mathbf{2_1}$	$\mathbf{1_4}$
$\mathbf{7_3}$	*8*	$\mathbf{9_3}$	$\mathbf{10_2}$	$\mathbf{11_3}$	$\mathbf{12_2}$	$\mathbf{13_4}$	$\mathbf{14_2}$	$\mathbf{1_4}$	$\mathbf{2_2}$	$\mathbf{3_4}$	$\mathbf{4_2}$
8	$\mathbf{7_3}$	$\mathbf{10_2}$	$\mathbf{9_3}$	$\mathbf{12_2}$	$\mathbf{11_3}$	$\mathbf{14_2}$	$\mathbf{13_4}$	$\mathbf{2_2}$	$\mathbf{1_4}$	$\mathbf{4_2}$	$\mathbf{3_4}$
$\mathbf{9_3}$	*10*	$\mathbf{11_3}$	$\mathbf{12_2}$	$\mathbf{13_3}$	$\mathbf{14_2}$	$\mathbf{1_4}$	$\mathbf{2_2}$	$\mathbf{3_4}$	$\mathbf{4_2}$	$\mathbf{5_4}$	$\mathbf{6_2}$
10	$\mathbf{9_3}$	$\mathbf{12_2}$	$\mathbf{11_3}$	$\mathbf{14_2}$	$\mathbf{13_3}$	$\mathbf{2_2}$	$\mathbf{1_4}$	$\mathbf{4_2}$	$\mathbf{3_4}$	$\mathbf{6_2}$	$\mathbf{5_4}$
$\mathbf{11_3}$	*12*	$\mathbf{13_3}$	$\mathbf{14_2}$	$\mathbf{1_3}$	$\mathbf{2_2}$	$\mathbf{3_4}$	$\mathbf{4_2}$	$\mathbf{5_4}$	$\mathbf{6_2}$	$\mathbf{7_4}$	$\mathbf{8_2}$
12	$\mathbf{11_3}$	$\mathbf{14_2}$	$\mathbf{13_3}$	$\mathbf{2_2}$	$\mathbf{1_3}$	$\mathbf{4_2}$	$\mathbf{3_4}$	$\mathbf{6_2}$	$\mathbf{5_4}$	$\mathbf{8_2}$	$\mathbf{7_4}$

$1, 2, 3, 4 : 3M$

Fig. 25.

Example 24 Here we show how the complete tripartite graph $K(12, 12, 14)$ may be decomposed into 5-cycles[2]. Our latin rectangle L is as shown in Figure 25.

Now $t < 2r$ and $2r - t = 2s - t = 10$, so this illustrates Case 1 in the previous theorem. Entries in italics are used by trades of type $1A$(ac) and remaining unsubscripted entries are used by trades of type $1A$(dwn).

6 Concluding remarks

The results from this paper, together with those from [7] and [4], give the following.

Theorem 25 *The complete tripartite graph $K(r, s, t)$ (with $r \leq s \leq t$) decomposes into 5-cycles only if r, s and t are either all odd or all even, 5 divides $rs + st + rt$ and $t \leq 4rs/(r + s)$. These necessary conditions are sufficient in the case when two partite sets have equal size or in the case when all partite sets have even size.*

[2] A decomposition of $K(12, 12, 14)$ into 5-cycles is in fact given in [7]. We include this example for illustrative purposes.

Thus the case when all partite sets are odd must be completed in order to give general necessary and sufficient conditions for a decomposition of a complete tripartite graph into 5-cycles. The techniques used in this paper may prove useful in solving this remaining case.

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