Decompositions of complete tripartite graphs into triangles with an edge attached

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Abstract

Let $K(r,s,t)$ denote the complete tripartite graph with partite sets of size r, s and t, where $r \leq s \leq t$. Let D be the graph consisting of a triangle with an edge attached. We show that $K(r,s,t)$ may be decomposed into copies of D if and only if 4 divides $rs+st+rt$ and $t \leq 3rs/(r+s)$.

1 Introduction

A graph with vertex set V is said to be a *complete n-partite* graph if V may be partitioned into n disjoint non-empty sets $V_1, V_2, \ldots, V_n$ (called *partite sets*) such that there exists exactly one edge between vertices from different partite sets, and no other edges. If $|V_i| = a_i$ for $1 \leq i \leq n$, this graph is denoted by $K(a_1, a_2, \ldots, a_n)$. We denote the graph consisting of a triangle with an attached edge by D. This is consistent with the notation used in [1] to classify graphs with at most four edges.

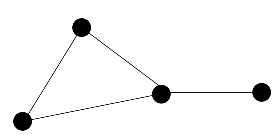


Figure 1: The graph D

*Work based on part of M.Sc. Thesis [4].

The problem of finding necessary and sufficient conditions to decompose complete n-partite graphs into copies of a subgraph H has been considered for many values n and graphs H.

For example, Hoffman and Liatti [6] solved the case when $n = 2$ and H is a smaller, complete bipartite graph. Necessary and sufficient conditions to decompose $K(r, s)$ into paths of fixed length are given by C.A. Parker in [8]. Sotteau ([9]) completely solved the bipartite case when H is a cycle.

For $n = 3$, Cavenagh [3] showed that $K(m, m, m)$ can be decomposed into k-cycles if and only if $k \leq 3m$ and k divides $3m^2$. The problem of decomposing complete tripartite graphs into 5-cycles has been considered in [7] and [5].

Billington [2] gave necessary and sufficient conditions for existence of a decomposition of any complete tripartite graph into specified numbers of 3-cycles and 4-cycles.

In Section 2 we give necessary conditions to decompose a complete tripartite graph into copies of D. We then construct representations of complete tripartite graphs called **latin representations** in Section 3. These are then used in Section 4 to give decompositions of complete tripartite graphs into copies of D.

The techniques used in this paper are very similar to those used in [5] to decompose complete tripartite graphs into 5-cycles, which in turn have their genesis in [2].

2 Necessary Conditions

Here we give some necessary conditions for the decomposition of a complete tripartite graph into copies of D.

THEOREM 2.1 *The complete tripartite graph $K(r, s, t)$ with $r \leq s \leq t$ decomposes into copies of the graph D only if*

1. $4 | rs + st + rt$ *and*
2. $t \leq 3rs/(r + s)$.

Proof

The first necessary condition follows from the fact that D has four edges, and so 4 must divide the number of edges in $K(r, s, t)$. For the second condition, observe that there are at most rs edge-disjoint triangles in $K(r, s, t)$ and that D contains a triangle. Therefore the total number of copies of D is less than or equal to rs, or equivalently $(rs + st + rt)/4 \leq rs$. The result follows. ∎

COROLLARY 2.2 *The complete tripartite graph $K(r,s,t)$ with $r \le s \le t$ decomposes into copies of D only if $t \le 2r$, $s \le 2r$ and $2t \le 3s$.*

Proof

Since $t \le 3rs/(r+s) \le 3rs/(r+r) = 3s/2$ we have $2t \le 3s$. Then from $s \le t \le 3rs/(r+s)$ we have $r+s \le 3r$ or $s \le 2r$. Finally, $t \le 3rs/(r+s) \le 3rs/(3s/2) = 2r$. $\blacksquare$

The following corollary follows from the first necessary condition given in Theorem 2.1.

COROLLARY 2.3 *The complete tripartite graph $K(r,s,t)$ with $r \le s \le t$ decomposes into copies of the graph D only if at least two partite sets are congruent modulo 4.*

Proof

Let $r \equiv r'$ (mod 4), $s \equiv s'$ (mod 4) and $t \equiv t'$ (mod 4), where $0 \le r', s', t' \le 3$. Then the triple (r', s', t') must belong to the following list:

$$\begin{array}{lllllll} (0,0,0), & (0,0,1), & (0,0,2), & (0,0,3), & (0,1,0), & (0,2,0), & (0,3,0), \\ (1,0,0), & (2,0,0), & (3,0,0), & (2,2,0), & (2,2,1), & (2,2,2), & (2,2,3), \\ (2,0,2), & (2,1,2), & (2,3,2), & (0,2,2), & (1,2,2), & (3,2,2). & \end{array}$$

This follows from the fact that $rs + rt + st$ is divisible by 4. $\blacksquare$

From the previous corollary, if $K(r,s,t)$ decomposes into copies of D then one of the following conditions must hold:

1. $r \equiv t \equiv 0$ (modulo 4) and s is odd, or $s \equiv t \equiv 2$ (modulo 4) and r is odd.

2. $s \equiv t \equiv 0$ (modulo 4) and r is odd, or $r \equiv t \equiv 2$ (modulo 4) and s is odd.

3. r and s are even.

We use this fact in Section 4 to split our proof into these three cases.

3 Latin representations

It is well-known that a latin square of order m is equivalent to a decomposition of the complete tripartite graph $K(m,m,m)$ into triangles. In this section we extend the idea of a latin square with the goal of decomposing complete tripartite graphs into copies of D.

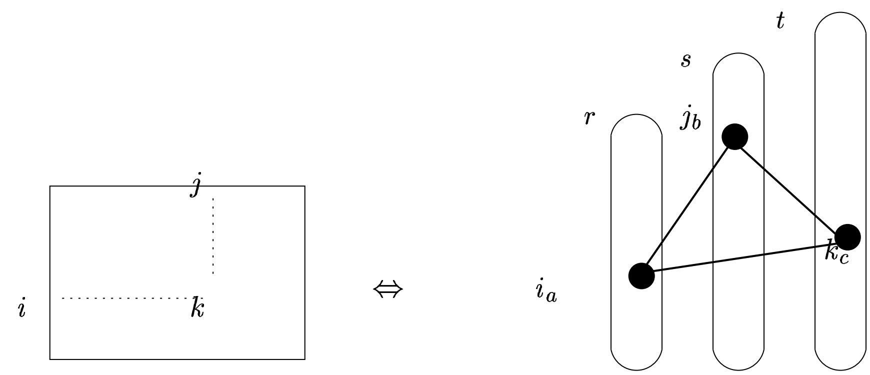
DEFINITION 3.4 *Consider a rectangular array of integers of order $r \times s$ with entries from the set $T = \{1, 2, \ldots, t\}$. If each entry appears at most once in each row and at most once in each column, we call such an array a* latin rectangle *of order $r \times s$ based on t elements.*

LEMMA 3.5 *Let r, s and t be integers such that $r \leq s \leq t$. A latin rectangle of order $r \times s$ based on t elements is equivalent to the existence of rs edge-disjoint triangles sitting inside the complete tripartite graph $K(r, s, t)$.*

Proof
First, label the vertices of $K(r, s, t)$ as follows:

$$\{1_a, 2_a, \ldots, r_a\} \cup \{1_b, 2_b, \ldots, s_b\} \cup \{1_c, 2_c, \ldots, t_c\}.$$

Then take a latin rectangle of order $r \times s$ based on t elements, with rows labelled from 1 to r and columns labelled from 1 to s. For each entry k in row i and column j we take a 3-cycle (i_a, j_b, k_c) in our tripartite graph. This can be seen more clearly in Figure 2.



Entry in latin rectangle Triangle in $K(r, s, t)$

Figure 2

Since no entry appears more than once in any row or column of a latin rectangle, the set of all such 3-cycles is edge-disjoint. ∎

So an $r \times s$ latin rectangle based on t elements can be thought of as a representation of rs triangles in the complete partite graph $K(r, s, t)$. Of course not all edges of $K(r, s, t)$ are represented in these rs triangles (unless $r = s = t$). However we can in fact extend the latin rectangle so that *every* edge in $K(r, s, t)$ is represented.

DEFINITION 3.6 *Let r, s and t be integers such that $r \leq s \leq t$. A* latin representation *of the complete tripartite graph $K(r, s, t)$ is a latin rectangle of order $r \times s$ based on t elements, together with a set of $t - s$ entries at the end of each row and a set of $t - r$ entries at the bottom of every column so that each entry from the set $T = \{1, 2, \ldots, t\}$ occurs once in each row and once in each column.*

So to construct a latin representation of the complete tripartite graph $K(r, s, t)$ we first take a latin rectangle of order $r \times s$ based on t elements. We then adjoin to the end of each row (column) any elements from the set T not already used in that row (column).

Each entry at the end of a row represents a single edge from the partite set of size r to the partite set of size t. For example, the entry k at the end of row i represents the edge $\{i_a, k_c\}$. Similarly, each entry from the bottom of a column represents a single edge from the partite set of size s to the partite set of size t. So a latin representation of $K(r, s, t)$ is in fact equivalent to a decomposition of $K(r, s, t)$ into rs triangles and $r(t - s) + s(t - r)$ single edges.

EXAMPLE 3.7 Figure 3 is a latin representation of the graph $K(4, 4, 5)$. We may think of this as a decomposition of $K(4, 4, 5)$ into 16 triangles and 8 further edges. For clarification we always use a double line to separate entries within the latin rectangle from entries outside the latin rectangle.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	

Figure 3

Certain kinds of latin representations will prove invaluable in finding decompositions of the complete tripartite graph $K(r, s, t)$ into copies of D. Our idea is to *trade* sets of triangles and other edges in $K(r, s, t)$ with copies of D, using our latin representation to keep a record of such exchanges, until all edges of $K(r, s, t)$ are used. This is a slight variation on the normal idea of a trade (see [10] for instance), so for the purpose of this paper we make the following definition.

DEFINITION 3.8 *Let M be a latin representation of the complete tripartite graph $K(r, s, t)$. A* trade *is a set of entries in M, corresponding to a set of triangles and edges in $K(r, s, t)$ which can be decomposed into copies of D.*

We define a *relabelling* of the entries of a trade to be a bijection ϕ from the set of entries $T = \{1, 2, \ldots, t\}$ onto itself. Thus every occurrence of $i \in T$ in the trade is replaced by $\phi(i)$. The relabelling of the entries in a trade does not change the structure of the corresponding set of edges in $K(r, s, t)$, and we can still decompose these edges into copies of D. So for every trade listed here, any relabelling of entries is permissible. We define the *transpose* of a trade to be the new trade formed by exchanging rows with columns. Where we refer to a particular trade type, we may sometimes mean the transpose — we do not always distinguish between them.

In a latin representation of a complete tripartite graph, D is clearly equivalent to an entry k in cell (i, j) of the latin rectangle, together with one entry from either the bottom of column j or the end of row i. The exchange of such two entries with D will be denoted as a trade of type 1.

It is also possible to exchange four triangles with three copies of D in a variety of ways. We use two such methods, giving rise to the trades shown in Figures 4 and 5.

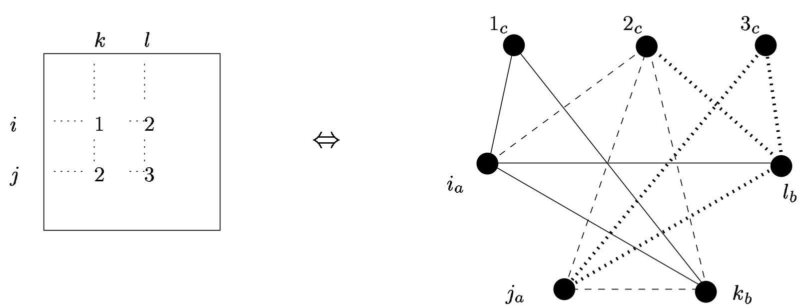


Figure 4: Trade type 2

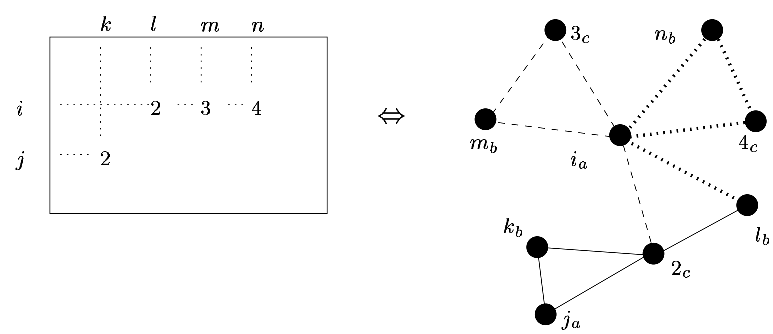


Figure 5: Trade type 3

Now that a variety of trades has been defined, we are ready to use them to construct decompositions of complete tripartite graphs into copies of D. Given a complete tripartite graph (that satisfies certain necessary conditions) it is sufficient to find a set of trades from the list mentioned that covers every entry in our latin representation, making sure no two trades overlap. Our task, then, is somewhat like a jigsaw puzzle, where our "pieces" are trades and the "picture" is a latin representation.

4 Decompositions

Here we show that the necessary conditions given in Section 2 are sufficient to give a decomposition of $K(r, s, t)$ into copies of D. Our latin representation used for each decomposition is as follows. Let L be the latin rectangle formed by taking the first r rows of the back circulant latin square of order s. We then adjoin to the end of rows 1 to r the set of integers $\{s+1, s+2, \ldots, t\}$ and adjoin to the ends of each column those integers from the set $\{1, 2, \ldots, t\}$ not already used in that column. Observe how trades of type 1, 2 and 3 occur many times in our latin representation.

For the next theorem we introduce a new trade.

1_1	2_2	3_2	4_2
2_2	3_3	4_4	5_5

Trade type 4

Figure 6

Entries with subscripts $1, 3, 4$ and 5 are used by trades of type 1 using one entry from the bottom of each column. Entries with subscript 2 are used by one trade of type 3.

In the following theorems we shall denote trades of type 1 that use entries from the ends of the rows as trades of type 1(ac), while trades of type 1(dwn) use entries from the bottoms of the columns.

THEOREM 4.9 *Let $K(r, s, t)$ be a complete tripartite graph with $r \leq s \leq t$ and either s is odd and $r \equiv t \equiv 0$ (modulo 4) or r is odd and $t \equiv s \equiv 2$ (modulo 4). Then $K(r, s, t)$ decomposes into copies of D if and only if the necessary conditions given in Theorem 2.1 hold.*

Proof

Note that in either case $2r - t$ is divisible by 4.

We first consider when $t = 2r$. From Theorem 2.1, we know that $t \leq 3rs/(r+s)$, or equivalently $r(t-s) \leq (2r-t)s$. So if $t = 2r$, $t = s$, and there are no entries at the ends of the rows. There are r entries at the end of each column, which may be used by trades of type 1(dwn), using all entries within the latin rectangle. This completes the decomposition in the case $t = 2r$.

Otherwise $t > 2r$. In each column we wish to use $t-r$ entries with trades of type 1(dwn) to use all entries from the bottom of the columns. This leaves $2r - t$ entries in each column. Working from the leftmost column, we use these remaining entries with trades of type 1(ac) until all entries from the ends of the rows are used. There are $r(t-s)$ entries at the ends of the rows, so we shall need at least

$$\frac{r(t-s)}{2r-t}$$

columns.

So let $r(t-s) = (2r-t)f + g$, where f and g are non-negative integers such that $g < 2r - t$. Since either r or $t - s$ is divisible by 4 we must have g is divisible by 4. Also, since $r(t-s) \leq (2r-t)s$, f is no greater than s.

The following set of cells is used by trades of type 1(dwn), using all $t-r$ entries from the bottoms of columns 1 to f:

$$\{(i,j) \mid 1 \leq j \leq f, \quad (j-1)(t-r)+1 \leq i \leq j(t-r)\},$$

where i is calculated modulo r.

This process can be seen more clearly in the following diagram of columns 1 to f:

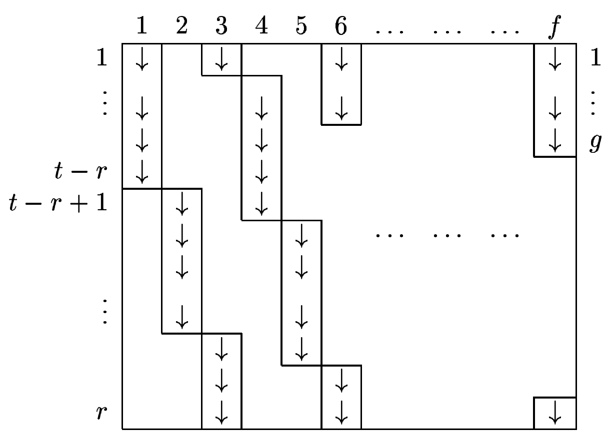


Figure 7

In each column $t-r$ entries are used by trades of type 1(dwn).

First consider when $f = s$. Then we must have $r(t-s) = (2r-t)s$, but there are $(2r-t)s$ entries remaining in our latin rectangle and $r(t-s)$

entries to be used from the ends of the rows. Thus we use all remaining entries with trades of type 1(ac).

We next consider when $s - f = 1$. Here the remaining entries of the first $f - 1$ columns are used by trades of type 1(ac). The final two columns are used as shown in Figure 8.

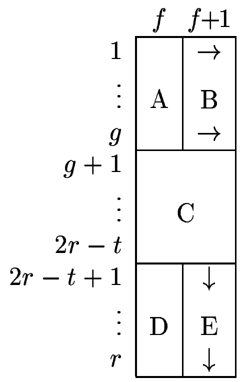


Figure 8

The arrows indicate when trades of type (ac) or (dwn) are being used.

The entries of B are used by trades of type 1(ac), while the entries of E are used by trades of type 1(dwn), using all entries from the bottom of column $f+1$. Since $2r-t-g$ is divisible by 4, the entries of C are used by trades of type 4 (in transpose). Any entries in A and D not already used by trades of type 1(dwn) are used by trades of type 1(ac).

For all remaining cases the entries of the first f columns not already used by trades of type 1(dwn) are used by trades of type 1(ac).

We next consider when $s - f = 2$. Then the final two columns are used as shown in Figure 9.

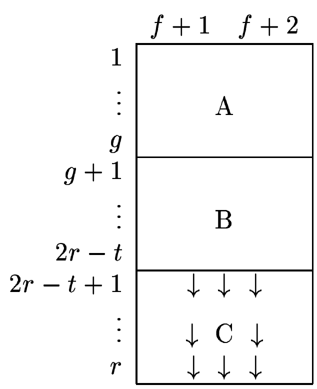


Figure 9

The entries of C are used by trades of type 1(dwn), using all entries from the bottom of columns $f+1$ and $f+2$. Since g is divisible by 4, the entries of A may be taken up by trades of type 4 (in transpose). The entries of B are used by trades of type 2.

Finally consider the case $s-f>2$. We use the entries in the final $s-f$ columns as shown in Figure 10.

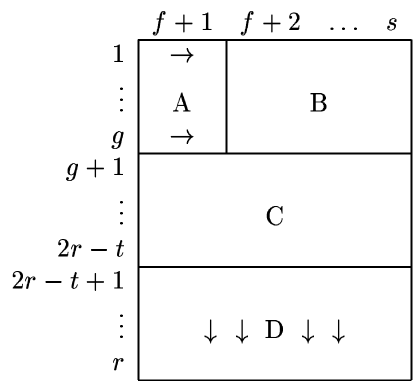


Figure 10

The entries of D are used by trades of type 1(dwn), using all entries from the bottom of the final $s-f$ columns. The entries of A are taken up by trades of type 1(ac). The entries of B and C are used by trades of type 2.

This completes all cases. ∎

THEOREM 4.10 *Let $K(r,s,t)$ be a complete tripartite graph with $r \le s \le t$ and either r is odd and $s \equiv t \equiv 0$ (modulo 4) or s is odd and $t \equiv r \equiv 2$ (modulo 4). Then $K(r,s,t)$ decomposes into copies of D if and only if the necessary conditions given in Theorem 2.1 hold.*

Proof

Note that in either case $2s-t$ is divisible by 4.

Let $s(t-r)=(2s-t)f+g$, where f and g are non-negative integers such that $g<2s-t$. Since either s or $t-r$ is divisible by 4 we must have that g is divisible by 4. Also, since $t \le 3rs/(r+s)$ (see Theorem 2.1) it follows that $s(t-r) \le (2s-t)r$, so f is at most r. Note that the case $t=2s$ is impossible, as this implies $r=t$, which contradicts $r \le s \le t$.

The proof of this theorem continues identically to Theorem 4.9, except in transpose. ∎

We next consider when both r and s are even. Our latin rectangle may be partitioned into 2×2 subsquares. In Lemma 4.11 we construct an array of pairs of integers that will be replaced by 2×2 subsquares in Theorem 4.13. Each pair of integers will indicate what type of trade is to be placed on each 2×2 subsquare.

LEMMA 4.11 *Let a, b, m and n be positive integers such that $m \leq n$, $b \leq a$, $mb + na \leq 2mn$ and $mb + na$ is even.*

Then we can construct an $m \times n$ array with the ordered pair of integers (x_{ij}, y_{ij}) in each cell (i, j) such that the following conditions hold for all pairs i, j, where $1 \leq i \leq m$ and $1 \leq j \leq n$.

1. $x_{ij}, y_{ij} \in \{0, 1, 2\}$.
2. $x_{ij} + y_{ij} \leq 2$.
3. $\sum_{i=1}^{m} x_{ij} = a$.
4. $\sum_{j=1}^{n} y_{ij} = b$.
5. $(x_{ij}, y_{ij}) \neq (0, 1)$.
6. *If $(x_{ij}, y_{ij}) = (1, 0)$ then $i = m$. There is an even number of such pairs and these pairs occur in adjacent columns.*

Proof

First consider when $m < a \leq 2m$. If $a = 2m$, the condition $mb + na \leq 2mn$ implies that $b = 0$. In this case we place the ordered pair $(2, 0)$ in every cell of the array. Otherwise $a < 2m$.

Let $mb = (2m - a)f + g$, where f and g are non-negative integers such that $g < 2m - a$. Since $mb + na \leq 2mn$, we have that $f \leq n$.

Assume that $f = n$. Then $g = 0$. Let S be the following set of cells:

$$\{(i, j) \mid 1 \leq j \leq n, \quad (j-1)(a-m) + 1 \leq i \leq j(a-m)\},$$

where i is calculated modulo m. If $(i, j) \in S$, we set x_{ij} to be 2 and y_{ij} to be 0. In every cell not belonging to S we set x_{ij} to be 1 and $y_{ij} = 1$. There are $(2m - a)n$ entries in our array not in S. Since $(2m - a)n = mb$ and S is spread evenly across the rows, the entries y_{ij} add up to b in each row. Conditions 1 to 6 clearly all hold in this case.

Otherwise we assume that $f < n$. Let S be the following set of cells:

$$\{(i, j) \mid 1 \leq j \leq f + 1, \quad (j-1)(a-m) + 1 \leq i \leq j(a-m)\}$$
$$\cup\{(i, j) \mid f + 2 \leq j \leq n, \quad 1 \leq i \leq (a - m)\},$$

where i is calculated modulo m. If $(i, j) \in S$, we set x_{ij} to be 2 and y_{ij} to be 0. For remaining cells in the first f columns we set $x_{ij} = y_{ij} = 1$. For rows 1 to g of column $f+1$ we also set $x_{ij} = y_{ij} = 1$. For all remaining cells in our array we set $x_{ij} = 1$ and $y_{ij} = 0$.

This can be seen more clearly in Figure 11. Entries in cells from S are in bold.

	1	2	$\dots$	f	$f+1$	$f+2$	$\dots$	n
1	$(\mathbf{2},\mathbf{0})$	$(1,1)$		$(1,1)$	$(1,1)$	$(\mathbf{2},\mathbf{0})$		$(\mathbf{2},\mathbf{0})$
$\vdots$	$(\mathbf{2},\mathbf{0})$	$(1,1)$		$(\mathbf{2},\mathbf{0})$	$(1,1)$	$(\mathbf{2},\mathbf{0})$		$(\mathbf{2},\mathbf{0})$
	$\vdots$	$\vdots$		$(\mathbf{2},\mathbf{0})$	$\vdots$	$\vdots$		$\vdots$
$a-m$	$(\mathbf{2},\mathbf{0})$	$(1,1)$		$\vdots$		$(\mathbf{2},\mathbf{0})$		$(\mathbf{2},\mathbf{0})$
$a-m+1$	$(1,1)$	$(\mathbf{2},\mathbf{0})$		$(\mathbf{2},\mathbf{0})$	$(1,1)$	$(1,0)$		$(1,0)$
	$(1,1)$	$(\mathbf{2},\mathbf{0})$	$\dots$	$(1,1)$	$(\mathbf{2},\mathbf{0})$	$(1,0)$	$\dots$	$(1,0)$
		$\vdots$		$(1,1)$	$(\mathbf{2},\mathbf{0})$			
$\vdots$	$\vdots$	$(\mathbf{2},\mathbf{0})$			$\vdots$			
		$(1,1)$		$\vdots$	$(\mathbf{2},\mathbf{0})$	$\vdots$		$\vdots$
		$\vdots$			$(1,0)$			
m	$(1,1)$	$(1,1)$		$(1,1)$	$(1,0)$	$(1,0)$		$(1,0)$

Figure 11

Conditions 1 to 5 are satisfied. To satisfy condition 6 we must do some rearranging. Let (i,j) be a cell such that $x_{ij} = 1$ and $y_{ij} = 0$. If there exists $i' \neq i$ such that $x_{i'j} = 1$ and $y_{i'j} = 0$, redefine x_{ij} to be 2 and $x_{i'j}$ to be zero. Note that conditions 1 to 5 still hold. Thus in each column we may ensure that there is at most one cell with $x_{ij} = 1$ and $y_{ij} = 0$. Such a cell may only occur in columns $f+1$ to n. In each case we can ensure that a pair $(1,0)$ occurs only in row m and that all such pairs are adjacent. In all cells other than those containing $(1,0)$, $x_{ij} + y_{ij}$ is even. The sum of all of the entries in all the cells is $ma + nb$, which is even. Therefore the number of cells containing $(1,0)$ must be even. Thus condition 6 holds.

Next consider when $a \leq m$. If $a = 0$ then $b = 0$, a trivial case. Otherwise assume that $a > 0$. Let $mb = af + g$, where f and g are non-negative integers such that $g < a$. Since $mb \leq an$, we have that $f \leq n$.

Assume that $f = n$. Then $g = 0$. Let S be the following set of cells:

$$\{(i,j) \mid 1 \leq j \leq n, \quad (j-1)(m-a)+1 \leq i \leq j(m-a)\},$$

where i is calculated modulo m. If $(i,j) \in S$, we set $x_{ij} = y_{ij} = 0$. In every cell not belonging to S we set $x_{ij} = 1$ and $y_{ij} = 1$. Since $an = mb$ and S is spread evenly across the rows, the entries y_{ij} add up to b in each row. Conditions 1 to 6 clearly hold in this case.

Finally, we assume that $f < n$. Let S be the following set of cells:

$$\{(i,j) \mid 1 \le j \le f+1, (j-1)(m-a)+1 \le i \le j(m-a)\}$$
$$\cup\{(i,j) \mid f+2 \le j \le n, 1 \le i \le m-a\},$$

where i is calculated modulo m. If $(i,j) \in S$, we set $x_{ij} = y_{ij} = 0$. For the remaining cells in the first f columns we set $x_{ij} = y_{ij} = 1$. For rows 1 to g of column $f+1$ we also set $x_{ij} = y_{ij} = 1$. For all remaining cells in our array we set $x_{ij} = 1$ and $y_{ij} = 0$.

Conditions 1 to 5 are clearly satisfied here. To satisfy condition 6 we do the same process of rearranging as in the case when $a > m$. ∎

LEMMA 4.12 *The complete tripartite graph $K(2,2,2)$ may be decomposed into copies of D.*

Proof

Label the vertices of $K(2,2,2)$ with $\{a_0, a_1\} \cup \{b_0, b_1\} \cup \{c_0, c_1\}$, where a different base denotes a different partite set. We denote a copy of the graph D with edges $\{w,x\}$, $\{x,y\}$, $\{y,w\}$ and $\{y,z\}$ by $(w,x,y;z)$. Then the following copies of D form a decomposition of $K(2,2,2)$:

$$(a_0, b_0, c_0; a_1), (b_0, c_1, a_1; b_1), (a_0, c_1, b_1; c_0).$$

∎

THEOREM 4.13 *Let $K(r,s,t)$ be a complete tripartite graph with $r \le s \le t$ such that r and s are even. Then if the necessary conditions of Theorem 2.1 are satisfied, $K(r,s,t)$ can be decomposed into copies of D.*

Proof

Consider the latin rectangle L in our latin representation. Let P_{ij} be a 2×2 subsquare of L occupying the following set of cells:

$$\{(m,n) \mid 2i-1 \le m \le 2i, \quad 2j-1 \le n \le 2j\},$$

for each i and j where $1 \le i \le r/2$ and $1 \le j \le s/2$. The subsquares P_{ij} form a partition of our latin rectangle L. Each subsquare is of the form shown in Figure 12, with each entry calculated modulo s.

$2k-1$	$2k$
$2k$	$2k+1$

Figure 12

With the help of Lemma 4.11 we can now construct a decomposition. Let $r = 2m$, $s = 2n$, $a = t - r$ and $b = t - s$. From the necessary conditions

$t \leq 3rs/(r+s)$ (see Theorem 2.1) we see that $mb + an \leq 2mn$. Since $r \leq s$ we have $m \leq n$ and because $t - s \leq t - r$, we have $b \leq a$. From Corollary 2.3 we have that if t is odd, $r + s$ is divisible by 4. It follows that $mb + na$ is even.

Therefore there exists an array of cells, each with an entry (x_{ij}, y_{ij}), that satisfies the properties given in Lemma 4.11. We replace each entry (x_{ij}, y_{ij}) with the 2×2 subsquare P_{ij}. On the entries of P_{ij} we apply a trade that uses $2x_{ij}$ entries from the bottoms of columns and $2y_{ij}$ entries from the ends of the rows.

If $x_{ij} = y_{ij} = 0$, we use a trade of type 2. (This is not possible if $r = s = 2$. This case is done by Lemma 4.12.) If $x_{ij} = 2$ and $y_{ij} = 0$ we use all entries in the subsquare with trades of type 1(dwn). On the other hand if $x_{ij} = 0$ and $y_{ij} = 2$ we use all entries with trades of type 1(ac). If $x_{ij} = y_{ij} = 1$ we use the 2×2 subsquare shown in Figure 13.

1	*2*
2	3

Figure 13

The entries in italics are used by trades of type 1(ac), while the remaining two entries are used by trades of type 1(dwn).

There are no subsquares with $x_{ij} = 0$ and $y_{ij} = 1$. The subsquares with $x_{ij} = 1$ and $y_{ij} = 0$ all occur in row r, the total number of them is even and they are adjacent, so we may use the corresponding subsquares with trades of type 4.

Thus every entry in our latin decomposition is used in a trade, and our decomposition is complete. ∎

5 Conclusion

We have now given necessary and sufficient conditions for the decomposition of a complete tripartite graph into copies of D.

THEOREM 5.14 *The complete tripartite graph $K(r, s, t)$ with $r \leq s \leq t$ decomposes into copies of the graph D if and only if*

1. $4 | rs + st + rt$ *and*
2. $t \leq 3rs/(r+s)$.

The results suggest the technique may also prove handy for decompositions of complete tripartite graphs into graphs similar to D — for example, a triangle with two edges attached or a triangle with a path attached.

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