On decomposing complete tripartite graphs into 5-cycles

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Abstract

Let $K(r,s,t)$ denote the complete tripartite graph with partite sets of size r, s and t, where $r \leq s \leq t$. Necessary and sufficient conditions are given for decomposability of $K(r,s,t)$ into 5-cycles whenever (i) $r = s$, (ii) $s = t$, and (iii) both r and s are divisible by 10. This extends work done in 1995 by Mahmoodian and Mirzakhani.

1 Introduction

A graph with vertex set V is said to be a *complete n-partite* graph if V may be partitioned into n disjoint non-empty sets $V_1, V_2, \ldots, V_n$ (called *partite sets*) such that there exists exactly one edge between vertices from different partite sets, and no other edges. If $|V_i| = a_i$ for $1 \leq i \leq n$, this graph is denoted by $K(a_1, a_2, \ldots, a_n)$. A *k-cycle*, with k edges $x_i x_{i+1}$, $1 \leq i \leq k-1$, and $x_k x_1$, on k distinct vertices x_i, $1 \leq i \leq k$, is denoted by $(x_1, x_2, \ldots, x_k)$ or $(x_1, x_k, x_{k-1}, \ldots, x_3, x_2)$ or any cyclic shift of these.

The problem of finding necessary and sufficient conditions to decompose complete n-partite graphs into k-cycles has been considered for many values of n and k. The case $n = 2$ was completely solved by Sotteau ([7]). Clearly the case $n = 2$ forces the cycle length to be even; $n = 3$ is the smallest value which permits odd cycle length.

It has been shown that a complete n-partite graph with each partite set of size k decomposes into k-cycles if and only if both n and k are odd [4]. Necessary and sufficient conditions to decompose the same graph into hamiltonian cycles are given in [5]. In the case of complete tripartite graphs, Cavenagh [2] showed that $K(m,m,m)$ can be decomposed into k-cycles if and only if $k \leq 3m$ and k divides $3m^2$. Billington [1]

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gave necessary and sufficient conditions for existence of a decomposition of any complete tripartite graph into specified numbers of 3-cycles and 4-cycles; the techniques used in that paper are extended and applied here.

The problem of decomposing complete tripartite graphs into 5-cycles was first considered by Mahmoodian and Mirzakhani [6]. We extend the results found in [6] giving necessary and sufficient conditions for the decomposition of the complete tripartite graph $K(r,s,t)$ into 5-cycles for the cases when:

- two partite sets have the same size (Section 5);
- both r and s are divisible by 10 (Section 6).

To solve these cases, in Section 3 we develop a new way of representing complete tripartite graphs. This representation allows us to monitor *trades* of sets of triangles and other edges with 5-cycles. In Section 4 we classify these trades into various types, which are then used in Sections 5 and 6 to give the required decompositions.

2 Necessary conditions

Here we give some necessary conditions for the decomposition of $K(r,s,t)$ into 5-cycles, and develop some theorems for later use. The following result is from Mahmoodian and Mirzakhani [6].

THEOREM 2.1 *If the complete tripartite graph $K(r,s,t)$ (where $r \leq s \leq t$) can be decomposed into 5-cycles, then the following conditions are satisfied:*

(i) r, s *and* t *are either all even or all odd;*

(ii) $5|(rs+rt+st)$;

(iii) $t \leq 4rs/(r+s)$.

Proof
Condition (i) arises from the fact that the degree of each vertex must be even for a decomposition to exist. Condition (ii) simply states that the number of edges must be divisible by five. To prove (iii), first observe that each 5-cycle must use at least one edge between any two partite sets, in particular the partite sets of smallest sizes r and s. Therefore rs, the number of edges between the two smallest partite sets, must be greater than or equal to the total number of 5-cycles, $(rs+st+rt)/5$. The result follows. ∎

COROLLARY 2.2 *If the complete tripartite graph $K(r,s,t)$ (with $r \leq s \leq t$) can be decomposed into 5-cycles, then $t \leq 3r$, $s \leq 3r$ and $t \leq 2s$.*

Proof
¿From condition (iii) in the previous theorem we have that $s \leq t \leq 4rs/(r+s)$, so that $(r+s) \leq 4r$, from which we have $s \leq 3r$. Therefore $t \leq 4rs/(r+s) \leq 4rs/(4s/3) = 3r$. Also, $t \leq 4rs/(r+s) \leq 4rs/(2r) = 2s$. ∎

COROLLARY 2.3 *If the complete tripartite graph $K(r,s,t)$ (with $r \leq s \leq t$) can be decomposed into 5-cycles, then at least two of the three partite sets have sizes which are congruent modulo 5.*

Proof

Let $r \equiv r' \pmod 5$, $s \equiv s' \pmod 5$ and $t \equiv t' \pmod 5$, where $0 \leq r', s', t' \leq 4$. Then the triple (r', s', t') must belong to the following list:

$$\begin{array}{lllll}
(0,0,0), & (0,0,1), & (0,0,2), & (0,0,3), & (0,0,4), \\
(0,1,0), & (0,2,0), & (0,3,0), & (0,4,0), & (1,0,0), \\
(2,0,0), & (3,0,0), & (4,0,0), & (1,1,2), & (1,2,1), \\
(2,1,1), & (1,3,3), & (3,1,3), & (3,3,1), & (2,2,4), \\
(2,4,2), & (4,2,2), & (3,4,4), & (4,3,4), & (4,4,3).
\end{array}$$

This follows from the fact that $rs + rt + st$ is divisible by 5; it is straightforward to check that these are all the possible cases. ∎

The following theorem is the main result from Mahmoodian and Mirzakhani's paper [6].

THEOREM 2.4 *Let $K(r,s,t)$ be a complete tripartite graph such that $r = s \leq t$ or $r \leq s = t$ and the conditions of Theorem 2.1 are satisfied. Then $K(r,s,t)$ has a decomposition into 5-cycles except possibly when two partite sets have order divisible by 5, but the third partite set does not.*

The following lemma is an extension of Lemma 5.2 in Mahmoodian and Mirzakhani's paper [6]. In this lemma we relax the constraint $r \leq s \leq t$.

LEMMA 2.5 *The complete tripartite graph $K(r_1 + r_2 + \cdots + r_n, ns, nt)$ may be decomposed into copies of the graphs $K(r_1, s, t)$, $K(r_2, s, t)$, $\ldots$, $K(r_{n-1}, s, t)$ and $K(r_n, s, t)$.*

Proof

Let $r_1 + r_2 + \cdots + r_n = R$, $ns = S$ and $nt = T$. Label the vertices of the graph $K(r_1 + r_2 + \cdots + r_n, ns, nt)$ as follows.

$$\{1_a, 2_a, \ldots, R_a\} \cup \{1_b, 2_b, \ldots, S_b\} \cup \{1_c, 2_c, \ldots, T_c\}.$$

Let L be any latin square of order n. For each entry k in row i and column j of L we place a copy of the graph $K(r_i, s, t)$ on the vertices

$$\{x_a \mid r_1 + r_2 + \cdots + r_{i-1} + 1 \leq x \leq r_1 + r_2 + \cdots + r_i\} \cup$$

$$\{y_b \mid s(j-1) + 1 \leq y \leq sj\} \cup \{z_c \mid t(k-1) + 1 \leq z \leq tk\}.$$

¿From the properties of a latin square, this gives a decomposition of $K(r_1 + r_2 + \cdots + r_n, ns, nt)$ into the graphs $K(r_1, s, t)$, $K(r_2, s, t), \ldots, K(r_{n-1}, s, t)$ and $K(r_n, s, t)$. ∎

COROLLARY 2.6 *If the graphs $K(r_1, s, t)$, $K(r_2, s, t)$, ..., $K(r_n, s, t)$ each admit a decomposition into 5-cycles, then the graph $K(r_1 + r_2 + \cdots + r_n, ns, nt)$ admits a decomposition into 5-cycles.*

The following corollary follows from Lemma 2.5, with $r_1 = r_2 = \ldots = r_n = r$.

COROLLARY 2.7 *The complete tripartite graph $K(nr, ns, nt)$, where n is any positive integer, can be decomposed into copies of the complete tripartite graph $K(r, s, t)$.*

COROLLARY 2.8 *If $K(r, s, t)$ can be decomposed into 5-cycles, then $K(nr, ns, nt)$, where n is any positive integer, also admits a decomposition into 5-cycles.*

3 Matrix representations

We now develop a new way of representing a complete tripartite graph which in fact extends the idea of a latin square. This representation will be an important tool for constructing decompositions of complete tripartite graphs.

DEFINITION 3.9 *Consider a rectangular array of integers of order $r \times s$ with entries from the set $T = \{1, 2, \ldots, t\}$. If each entry appears at most once in each row and at most once in each column, we call such an array a* latin rectangle *of order $r \times s$ based on t elements.*

It is well-known that a latin square of order m is equivalent to a decomposition of the complete tripartite graph $K(m, m, m)$ into triangles. An interesting extension of this result is the following lemma.

LEMMA 3.10 *Let r, s and t be integers such that $r \leq s \leq t$. A latin rectangle of order $r \times s$ based on t elements is equivalent to the existence of rs edge-disjoint triangles sitting inside the complete tripartite graph $K(r, s, t)$.*

Proof
First, label the vertices of $K(r, s, t)$ as follows:

$$\{1_a, 2_a, \ldots, r_a\} \cup \{1_b, 2_b, \ldots, s_b\} \cup \{1_c, 2_c, \ldots, t_c\}.$$

Then take a latin rectangle of order $r \times s$ based on t elements, with rows labelled from 1 to r and columns labelled from 1 to s. For each entry k in row i and column j we take a 3-cycle (i_a, j_b, k_c) in our tripartite graph. This can be seen more clearly in Figure 3.1.

Since no entry appears more than once in any row or column of a latin rectangle, the set of all such 3-cycles is edge-disjoint. ∎

So an $r \times s$ latin rectangle based on t elements can be thought of as a representation of rs triangles in the complete partite graph $K(r, s, t)$. Of course not all edges of $K(r, s, t)$ are represented in these rs triangles (unless $r = s = t$). However we can in fact extend the latin rectangle so that *every* edge in $K(r, s, t)$ is represented.

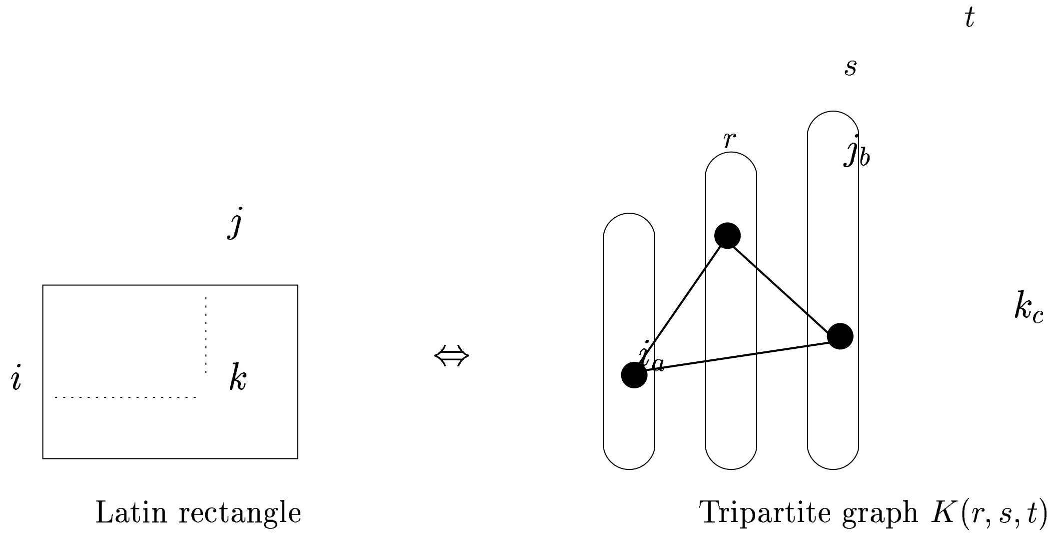


Figure 3.1

DEFINITION 3.11 *Let r, s and t be integers such that $r \leq s \leq t$. A* matrix representation *of the complete tripartite graph $K(r, s, t)$ is a latin rectangle of order $r \times s$ based on t elements, together with a set of $t - s$ entries at the end of each row and a set of $t - r$ entries at the bottom of every column so that each entry from the set $T = \{1, 2, \ldots, t\}$ occurs once in each row and once in each column.*

So to construct a matrix representation of the complete tripartite graph $K(r, s, t)$ we first take a latin rectangle of order $r \times s$ based on t elements. We then adjoin to the end of each row any elements from the set T not already used in that row (in any order). Finally, to the bottom of each column we adjoin any entries from the set T not already used in that column.

Each entry at the end of a row represents a single edge from the partite set of size r to the partite set of size t. For example, the entry k at the end of row i represents the edge $\{i_a, k_c\}$. Similarly, each entry from the bottom of a column represents a single edge from the partite set of size s to the partite set of size t. Thus the entry k at the bottom of column j represents the edge $\{j_b, k_c\}$. So a matrix representation of $K(r, s, t)$ is in fact equivalent to a decomposition of $K(r, s, t)$ into rs triangles and $r(t - s) + s(t - r)$ single edges.

EXAMPLE 3.12 Figure 3.2 is a matrix representation of the graph $K(5, 5, 7)$. We may think of this as a decomposition of $K(5, 5, 7)$ into 25 triangles and 20 further edges. For clarification we always use a double line to separate entries within the latin rectangle from entries outside the latin rectangle. Moreover, we stress that each entry inside the latin rectangle represents a triangle (*three* edges), whereas each entry outside the latin rectangle represents a single edge.

1	2	3	4	5	6 7
2	3	6	7	4	1 5
3	5	7	6	2	1 4
4	6	1	3	7	2 5
5	7	2	4	1	3 6
6 7	1 4	4 5	1 5	3 6	

Figure 3.2

Certain kinds of matrix representations will prove invaluable in finding decompositions of the complete tripartite graph $K(r,s,t)$ into 5-cycles. Our idea is to *trade* sets of triangles and other edges in $K(r,s,t)$ with 5-cycles, using our matrix representation to keep a record of such exchanges, until all edges of $K(r,s,t)$ are used. This is a slight variation on the normal idea of a trade (see [8] for instance), so for the purpose of this paper we make the following definition.

DEFINITION 3.13 *Let M be a matrix representation of the complete tripartite graph $K(r,s,t)$. A* trade *is a set of entries in M, corresponding to a set of triangles and edges in $K(r,s,t)$ which can be decomposed into 5-cycles.*

EXAMPLE 3.14 Consider the matrix representation of the complete tripartite graph $K(5,5,7)$ given in Example 3.12. In Figure 3.3 six entries of this matrix representation are in italics.

1	*2*	3	4	5	*6 7*
2	3	6	7	4	1 5
3	5	7	6	2	1 4
4	6	1	3	7	2 5
5	7	2	4	1	3 6
6 *7*	1 4	4 5	1 5	3 6	

Figure 3.3

These six entries correspond to the two triangles and four edges, shown on the left of Figure 3.4, the edges of which decompose into two 5-cycles, shown on the right of Figure 3.4. Thus these six entries in fact form what we refer to in this paper as a trade.

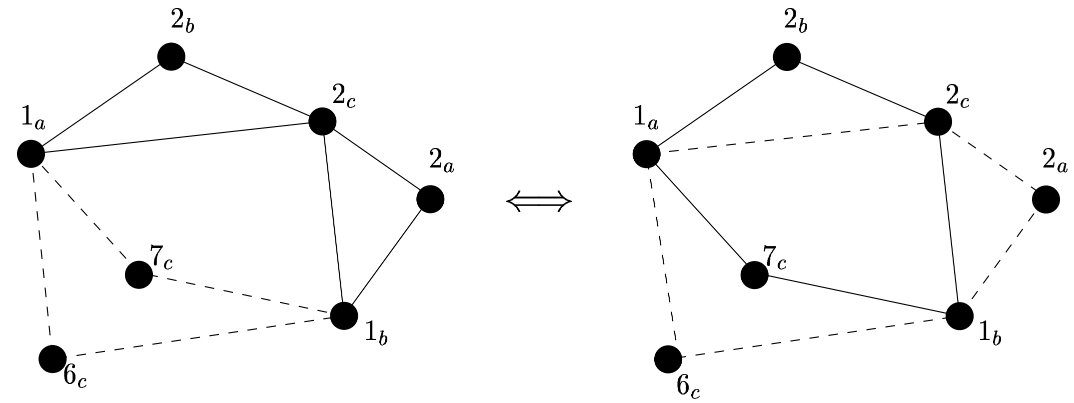


Figure 3.4

4 Classification of 5-cycle trades

In this section we find and classify a range of trades that we need subsequently.

Consider a matrix representation M (containing a latin rectangle L) of a complete tripartite graph $K(r,s,t)$. Recall that a *trade* is a set of entries in M, corresponding to edges in $K(r,s,t)$ which are decomposable into 5-cycles. We shall use two types of trades. In the first type, entries from both inside and outside our latin rectangle L are used. In this type of trade we are exchanging a set of triangles and a set of edges with a set of 5-cycles. In the second type, no entries from outside the latin rectangle are used. In this case we are trading a set of triangles with a set of 5-cycles.

We define a *relabelling* of the entries of a trade to be a bijection ϕ from the set of entries $T = \{1, 2, \ldots, t\}$ onto itself. Thus every occurrence of $i \in T$ in the trade is replaced by $\phi(i)$. The relabelling of the entries in a trade does not change the structure of the corresponding set of edges in $K(r,s,t)$, and we can still decompose these edges into 5-cycles. So for every trade listed here, any relabelling of entries is permissible. In some trades we describe, rows and columns appear adjacent; note that this may not be the case in the matrix representation. We define the *transpose* of a trade to be the new trade formed by exchanging rows with columns. Where we refer to a particular trade type, we may sometimes mean the transpose — we do not always distinguish between them.

4.1 Trade type 1

Trades of type 1 involve exchanges of triangles *and* extra edges with 5-cycles. A trade of type 1 always uses twice as many entries from outside the latin rectangle L (in our matrix representation M) as inside L.

Figure 4.1 indicates how two 3-cycles ($(1,2,3)$ and $(3,4,6)$) and four edges (forming a 4-cycle $(1,5,4,7)$) may cover the same set of edges as two 5-cycles ($(1,2,3,4,5)$ and $(1,3,6,4,7)$).

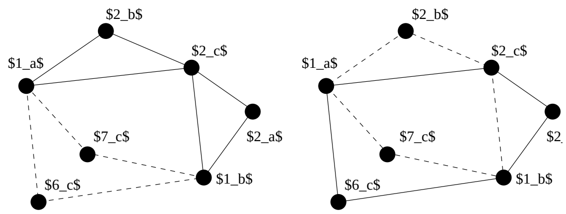


Figure 4.1

We label each vertex with a row (subscript a), column (subscript b), or entry (subscript c) from a complete tripartite graph, giving rise to four subtypes of trades, shown in Figures 4.2, 4.3, 4.4 and 4.5.

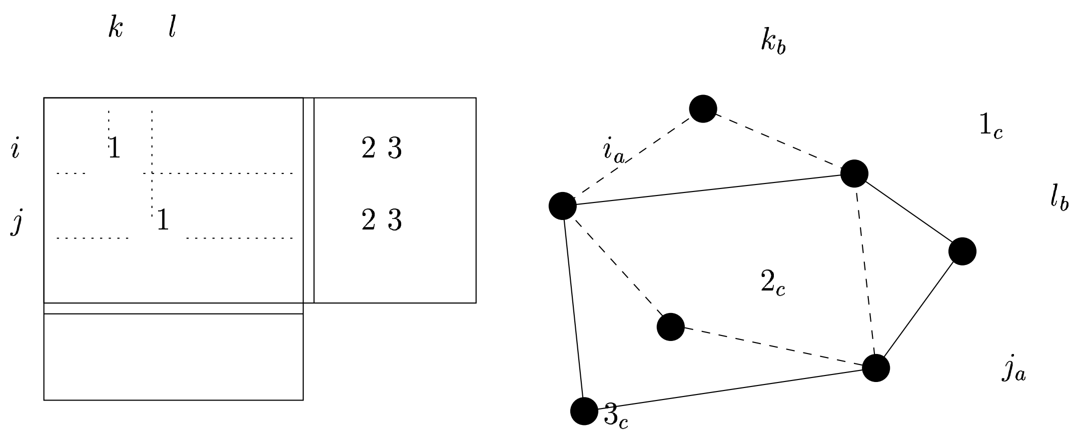


Figure 4.2: Trade type $1A$

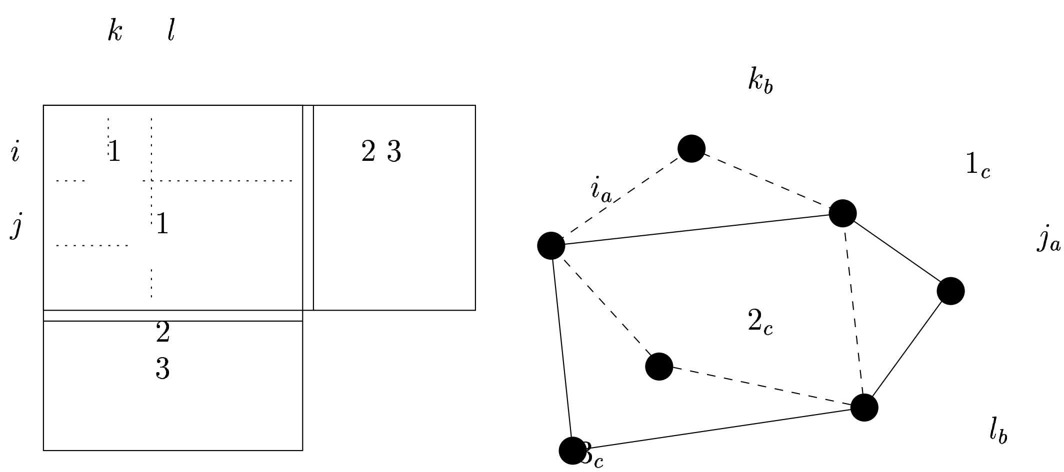


Figure 4.3: Trade type $1B$

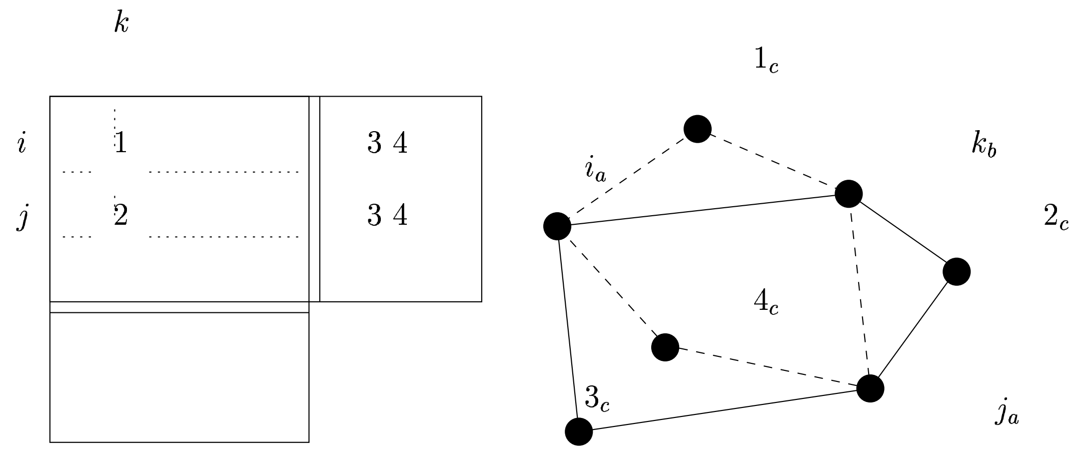


Figure 4.4: Trade type $1C$

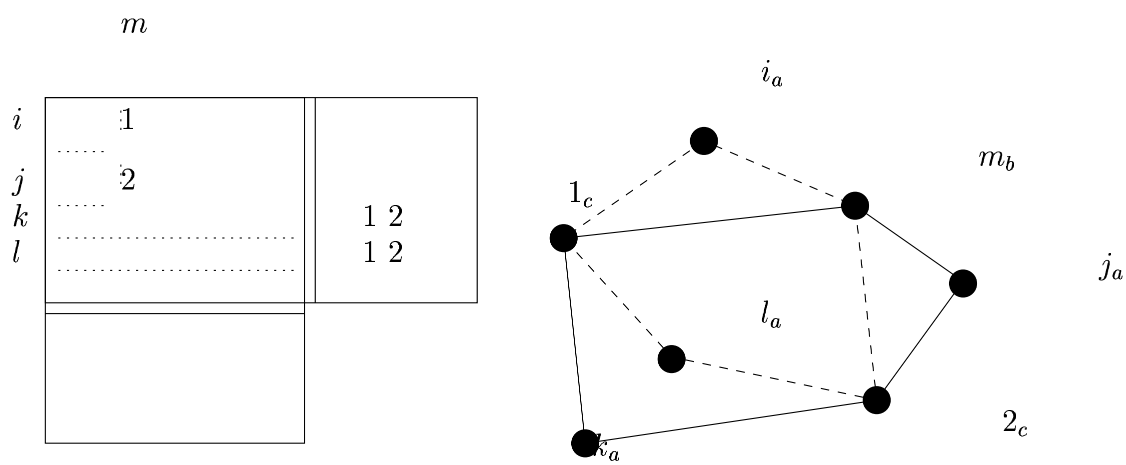


Figure 4.5: Trade type $1D$

We may also trade three triangles and six edges (which form a 6-cycle) with three 5-cycles, as shown in Figure 4.6.

Note that it is possible to replace 4 with 3 and 5 with 2 in the trade given in Figure 4.6.

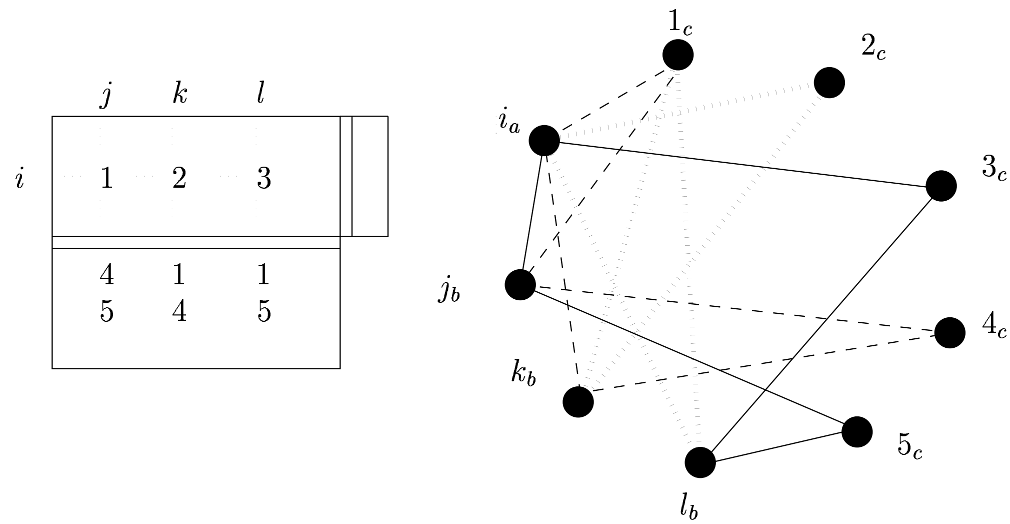
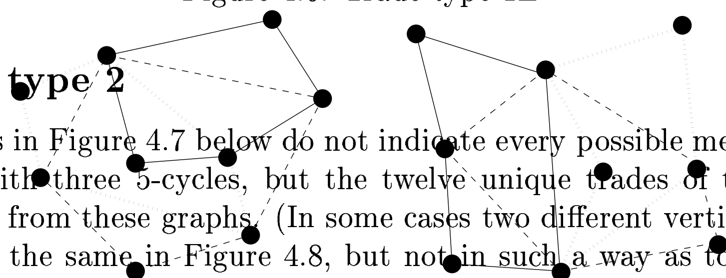


Figure 4.6: Trade type $1E$



4.2 Trade type 2

The four graphs in Figure 4.7 below do not indicate every possible method of trading five triangles with three 5-cycles, but the twelve unique trades of type 2 listed in Figure 4.8 arise from these graphs. (In some cases two different vertices from Figure 4.7 are labelled the same in Figure 4.8, but not in such a way as to allow multiple edges between vertices.)

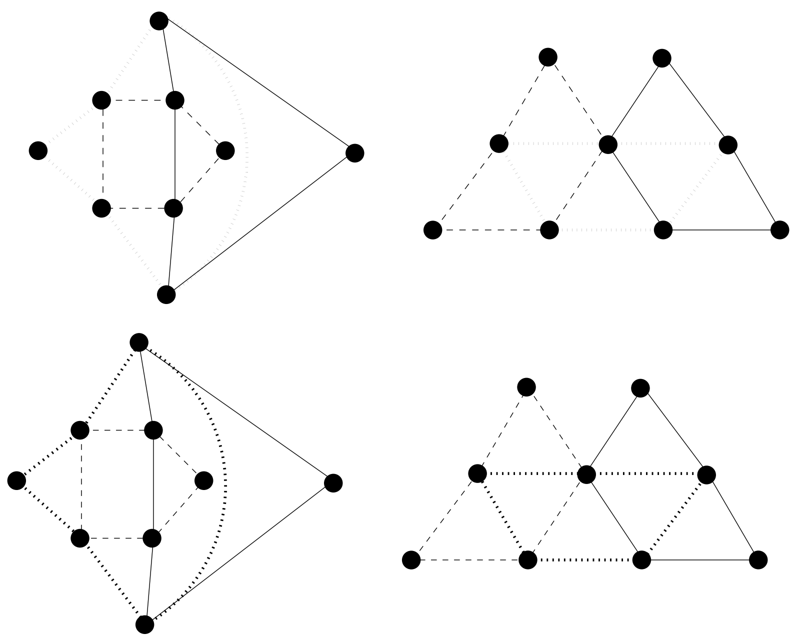


Figure 4.7

For each trade, recall that entries can be relabelled and the transpose can always be taken. In these trades in Figure 4.8, rows and columns appear to be adjacent, but this need not be the case within a particular matrix representation M. Note that these trades lie entirely within the latin rectangle L, and they do not involve the extra entries in M at the ends of the rows and columns of L.

1	2	3
4	3	

Trade type 2A

1	2	3
2	3	

Trade type 2B

1	2	3	
	3		1

Trade type 2C

1	2		
		1	
		2	1

Trade type 2D

1		2
	1	
	2	1

Trade type 2E

	1	
1	2	3
	3	

Trade type 2F

1	2	3
	1	
		1

Trade type 2G

1		2
	1	
	2	3

Trade type 2H

1	2	3
	3	
2		

Trade type 2I

1	2	
	1	2
		1

Trade type 2J

1	2	
	1	3
		1

Trade type 2K

	1	2
1		3
2		

Trade type 2L

Figure 4.8

This completes the list of all the small trades we shall be using. Later we shall introduce some larger trades constructed from these smaller ones.

Now that a variety of trades has been defined, we can use them to construct decompositions of complete tripartite graphs into 5-cycles. We saw earlier that each complete tripartite graph has a (not unique) *matrix representation*, which consists of a latin rectangle with unordered sets at the end of each row and column. Given a complete tripartite graph (satisfying certain necessary conditions), it is sufficient to find is a set of trades from the list mentioned that covers every entry in our matrix representation, making sure no two trades overlap. Our task, then, is somewhat like a jigsaw puzzle, where our "pieces" are trades and the "picture" is a matrix representation.

We see this process in action in the following lemma.

LEMMA 4.15 *The graphs $K(3,5,5)$, $K(5,5,5)$, $K(5,5,7)$ and $K(5,5,9)$ each admit decompositions into 5-cycles.*

Proof

For $K(5,5,9)$, Figure 4.9 gives a suitable matrix representation and indicates trades which yield a possible decomposition into 5-cycles. The subscript attached to each entry indicates the trade containing that entry; the key below indicates what type of trade each subscript corresponds to. The numbers in bold always belong to a trade of type 2. To see the trades more clearly in the diagram, the reader may wish to circle entries from different trades with differently coloured pens.

1	$\mathbf{2_1}$	*3*	4_3	5_4	6_3 7_3 8_4 9_4
$\mathbf{2_1}$	$\mathbf{3_1}$	$\mathbf{4_1}$	5_7	1_6	6_7 7_7 8_6 9_6
3	$\mathbf{4_1}$	*5*	1_5	2_2	6_2 7_2 8_5 9_5
4_3	5_7	1_5	*2*	*3*	6_3 7_3 8_5 9_5
5_4	1_6	2_2	*3*	*4*	6_2 7_2 8_4 9_4
6	6_7	*6*	*6*	*6*	
7	7_7	*7*	*7*	*7*	
8	8_6	*8*	*8*	*8*	
9	9_6	*9*	*9*	*9*	

Subscripts: $1 : 2F$ $\quad 2, 3, 4, 5 : 1A$ $\quad 6, 7 : 1B$

(Entries in italics are used by trades of type $1C$.)

Figure 4.9

A decomposition of $K(5, 5, 7)$ into 5-cycles is given in Figure 4.10.

1_1	$\mathbf{2_6}$	3_3	4_3	$\mathbf{5_6}$	6_1 7_1
2_1	3_2	$\mathbf{4_7}$	$\mathbf{5_6}$	$\mathbf{1_6}$	6_1 7_1
3_2	4_4	$\mathbf{5_7}$	$\mathbf{1_8}$	$\mathbf{2_6}$	6_4 7_4
$\mathbf{4_7}$	5_4	$\mathbf{1_8}$	$\mathbf{2_8}$	3_5	6_4 7_4
$\mathbf{5_7}$	$\mathbf{1_8}$	$\mathbf{2_8}$	3_5	$\mathbf{4_7}$	6_5 7_5
6_2	6_2	6_3	6_3	6_5	
7_2	7_2	7_3	7_3	7_5	

Subscripts: 1,3,4: $1C$ 2: $1A$ 5: $1B$ 6: $2I$ 7: $2D$ 8: $2J$

Figure 4.10

For a decomposition of the graph $K(5, 5, 5)$ into 5-cycles, see [6]; an alternative decomposition is given in Figure 4.11.

$\mathbf{1_1}$	$\mathbf{2_1}$	$\mathbf{3_1}$	$\mathbf{4_2}$	$\mathbf{5_2}$
$\mathbf{2_1}$	$\mathbf{3_1}$	$\mathbf{4_2}$	$\mathbf{5_2}$	$\mathbf{1_2}$
$\mathbf{3_3}$	$\mathbf{4_3}$	$\mathbf{5_3}$	$\mathbf{1_5}$	$\mathbf{2_5}$
$\mathbf{4_3}$	$\mathbf{5_4}$	$\mathbf{1_4}$	$\mathbf{2_4}$	$\mathbf{3_5}$
$\mathbf{5_3}$	$\mathbf{1_4}$	$\mathbf{2_4}$	$\mathbf{3_5}$	$\mathbf{4_5}$

Subscripts: 1,2,4,5: $2A$ 3: $2F$

Figure 4.11

Finally we give a decomposition of $K(3,5,5)$ into 5-cycles:

$\mathbf{1_1}$	$\mathbf{2_1}$	$\mathbf{3_1}$	$\mathbf{4_2}$	$\mathbf{5_2}$
$\mathbf{3_1}$	$\mathbf{1_1}$	$\mathbf{5_2}$	$\mathbf{2_2}$	$\mathbf{4_2}$
2_3	3_3	4_4	5_4	1_4
4_3 5_3	4_3 5_3	1_4 2_4	1_4 3_4	2_4 3_4

Subscripts: 1,2: $2B$ 3: $1C$ 4: $1E$

Figure 4.12

∎

5 Two partite sets equal

Here we consider the problem of decomposing the complete tripartite graph $K(r,s,t)$ (with $r \leq s \leq t$) into 5-cycles when two partite sets have equal size. We may assume that both of these sets have size divisible by 5, since other cases are done in [6]. We first consider the case $s = t$, irrespective of whether r, s and t are all odd or all even. We then solve the case $r = s$ with all partite sets of odd size. Finally, we consider the case $r = s$ with all parts of even size.

We now give some notation that will be used in the proof of our next result. Let a be a positive integer and M a set of real numbers. Then $a * M$ is defined to be the set formed from the sum of any a (possibly non-distinct) elements of M. For example, if $M = \{0,1,3\}$ and $a = 3$, then

$$3 * \{0,1,3\} = \{0,1,2,3,4,5,6,7,9\}.$$

We also define $M \oplus N$ (for sets M and N) to be the set of all possible sums of the form $m + n$ where m and n are elements of M and N respectively.

THEOREM 5.16 *Let $K(r,5S,5S)$ be a complete tripartite graph with $r \leq 5S$ that satisfies the necessary conditions of Theorem* 2.1. *Then there exists a decomposition of $K(r,5S,5S)$ into* 5*-cycles.*

Proof

For this proof we modify the structure of our matrix representation. Rather than using a latin rectangle of order $r \times 5S$ we shall use a latin *square* of order $5S$. Each entry in the first r rows corresponds to a triangle in the usual fashion. Each entry in the final $5S - r$ rows corresponds to one edge only. Thus we have simply ordered the entries from outside the latin rectangle into rows, so that the matrix representation may be thought of as a latin square.

We first need to define two new trades, both of which combine smaller trades of type 1 or 2. In the following trades, entries above the double line must occur in the first r rows, while entries below the double line must occur in the final $5S - r$ rows.

$\mathbf{1_1}$	$\mathbf{2_1}$	$\mathbf{3_1}$	5_2	4_2
$\mathbf{2_1}$	$\mathbf{3_1}$	4_6	1_6	5_6
3_3	1_3	5_5	4_5	2_5
6_7	7_7	8_7	9_4	10_4
4_2	5_2	2_5	2_5	3_5
5_2	4_2	1_6	3_5	1_6
8_3	9_3	7_7	10_6	6_4
9_3	8_3	10_6	6_4	7_4
7_7	10_7	6_7	8_6	9_5
10_7	6_7	9_5	7_4	8_6

Subscripts: 1: $2A$ 2: $1D$ 3,4: $1C$ 5,6,7: $1E$
Trade type $3A$
Figure 5.1

Note that each of the entries from 1 to 10 is used in every column.

1	2	3_1	4_1	5_1
6	7	8_2	9_2	10_2
15_7	11_3	12_4	13_5	14_6
12_4	15_7	14_6	11_3	13_5
11	12	13_8	14_8	15_8
3	4	5_1	2_2	1_2
4	3	1_2	5_1	2_2
5_7	1_3	2_4	3_5	4_6
2_4	5_7	4_6	1_3	3_5
8	9	10_2	7_1	6_1
9	8	6_1	10_2	7_1
10_7	6_3	7_4	8_5	9_6
7_4	10_7	9_6	6_3	8_5
13	14	15_8	12_8	11_8
14	13	11_8	15_8	12_8

Subscripts: 1,2,8: $1E$ 3,4,5,6,7: $1A$
Trade type $3B$
(Unsubscripted entries are used by trades of type $1C$.)
Figure 5.2

Note that all entries from 1 to 15 are used in each column. In Lemma 4.15 we showed that the graphs $K(3,5,5)$ and $K(5,5,5)$ can be decomposed into 5-cycles. Thus we may use entries in our latin square corresponding to these graphs as trades.

Consider the first five columns of a matrix representation of the graph $K(r, 5S, 5S)$. We show how to construct these columns and use every entry with trades. The first five columns will be made up of S 5×5 latin squares: L_1, $L_2, \ldots, L_S$, where L_i is based on the entries $\{5i-4, 5i-3, 5i-2, 5i-1, 5i\}$, for $1 \le i \le S$. Each latin square has some rows within the first r rows of our matrix representation and (possibly) some rows from the final $5S - r$ rows.

We deal with these latin squares in one of four ways:

1. Two 5×5 latin squares may be used with a trade of type $3A$, using six of the final $5S - r$ rows.

2. Three 5×5 latin squares may be used with a trade of type $3B$, using ten of the final $5S - r$ rows.

3. One 5×5 latin square may be used with the graph $K(5,5,5)$, using none of the final $5S - r$ rows.

4. One 5×5 latin square may be used with the graph $K(3,5,5)$, using two of the final $5S - r$ rows.

So by means of these four techniques, the number of entries used in the final $5S - r$ rows of each column may be any even integer from the following set:

$$S * \{10/3, 3, 2, 0\}.$$

Since $s/3 \le r \le s$ (see Corollary 2.2), we have $5S/3 \le r \le 5S$, or alternatively $0 \le 5S - r \le 10S/3$. Therefore the even integer $5S - r$ is an element of the above set for any permissible r. So in any case we can partition all edges represented by the first five columns into 5-cycles. To construct the next five columns, simply add 5 (modulo $5S$) to each entry in the first five columns, and apply exactly the same trades. In this fashion we can decompose the entire graph $K(r, 5S, 5S)$. ∎

EXAMPLE 5.17 Using Theorem 5.16, the complete tripartite graph $K(7, 15, 15)$ can be decomposed into 5-cycles. The matrix representation is as follows:

1_1	2_1	3_1	5_1	4_1	6_2	7_2	8_2	10_2	9_2	11_3	12_3	13_3	15_3	14_3
2_1	3_1	4_1	1_1	5_1	7_2	8_2	9_2	6_2	10_2	12_3	13_3	14_3	11_3	15_3
3_1	1_1	5_1	4_1	2_1	8_2	6_2	10_2	9_2	7_2	13_3	11_3	15_3	14_3	12_3
6_1	7_1	8_1	9_1	10_1	11_2	12_2	13_2	14_2	15_2	1_3	2_3	3_3	4_3	5_3
11_4	12_4	13_4	14_4	15_4	1_5	2_5	3_5	4_5	5_5	6_6	7_6	8_6	9_6	10_6
13_4	11_4	15_4	12_4	14_4	3_5	1_5	5_5	2_5	4_5	8_6	6_6	10_6	7_6	9_6
12_4	13_4	14_4	15_4	11_4	2_5	3_5	4_5	5_5	1_5	7_6	8_6	9_6	10_6	6_6
4_1	5_1	1_1	2_1	3_1	9_2	10_2	6_2	7_2	8_2	14_3	15_3	11_3	12_3	13_3
5_1	4_1	2_1	3_1	1_1	10_2	9_2	7_2	8_2	6_2	15_3	14_3	12_3	13_3	11_3
10_1	9_1	6_1	7_1	8_1	12_2	13_2	14_2	15_2	11_2	2_3	3_3	4_3	5_3	1_3
9_1	10_1	7_1	8_1	6_1	13_2	14_2	15_2	11_2	12_2	3_3	4_3	5_3	1_3	2_3
8_1	6_1	9_1	10_1	7_1	14_2	15_2	11_2	12_2	13_2	4_3	5_3	1_3	2_3	3_3
7_1	8_1	10_1	6_1	9_1	15_2	11_2	12_2	13_2	14_2	5_3	1_3	2_3	3_3	4_3
14_4	15_4	11_4	12_4	13_4	4_5	5_5	1_5	3_5	2_5	9_6	10_6	6_6	8_6	7_6
15_4	14_4	12_4	13_4	11_4	5_5	4_5	2_5	1_5	3_5	10_6	9_6	7_6	6_6	8_6

Subscripts: $1,2,3 : 3A$ $\quad 4,5,6 : K(3,5,5)$

Figure 5.3

THEOREM 5.18 *Let $K(5R, 5R, t)$ be a complete tripartite graph with $t \geq 5R$, R and t odd, that satisfies the necessary conditions given in Theorem 2.1. Then there exists a decomposition of $K(5R, 5R, t)$ into 5-cycles.*

Proof

¿From Corollary 2.2 we have $5R \leq t \leq 10R - 1$ (both R and t are odd).

The cases when $R = 1$ are done in Lemma 4.15. So now let $R = 3$. Then t can take odd integer values from 15 to 29. We can decompose $K(15, 15, 17)$ into 5-cycles using the fact that $K(5, 5, 5)$ and $K(5, 5, 7)$ decompose into 5-cycles, together with Corollary 2.6. Since $K(5, 5, 9)$ also decomposes into 5-cycles, we can similarly decompose $K(15, 15, t)$ into 5-cycles for any odd value of t except possibly 29, since

$$\{15, 17, 19, \ldots, 27\} \subseteq 3 * \{5, 7, 9\}.$$

To decompose $K(15, 15, 29)$, we first construct a matrix representation of this graph. Let L_5 be any latin square of order 5 and let I_3 be a symmetric latin square of order 3. The latin square $I_3 \times L_5$, of order 15, is the latin rectangle L to be used in our matrix representation M. At the end of each row and column we add the set of entries

$$\{16, 17, \ldots, 29\}.$$

Notice that our latin square is a 3×3 array of 5×5 latin squares and that we can group these 5×5 latin squares into three "diagonals" A, B, C of (possibly different) 5×5 squares, as seen in Figure 5.4.

A	B	C
C	A	B
B	C	A

Figure 5.4

The 5×5 subsquares in a diagonal are not necessarily identical.

Using the latin squares of type L_5 on diagonal A and the entries $\{16, 17, 18, 19\}$ at the end of each row and column we may place three copies of the graph $K(5, 5, 9)$ (see Figure 4.9). In this way we are using the entire graph $K(5, 5, 9)$ as a trade; we have shown already that it decomposes into 5-cycles. The remaining entries can be used up with trades of type $1B$, where each unused cell (i, j) in diagonal B is paired with the cell (j, i) in diagonal C, using a pair of entries from the end of row i and column i. Since I_3 is symmetric, the cell (i, j) contains the same entry as cell (j, i). We are free to choose *any* pair from the end of row i or column i. Since there are five entries in each row that each use a pair of entries from the end of that row, all ten entries $\{20, 21, \ldots, 29\}$ are used at the end of every row. Similarly, the entries $\{20, 21, \ldots, 29\}$ are used at the bottom of each column. This completes the decomposition.

Now consider the general graph $K(5R, 5R, t)$, with R odd, $R \geq 5$ and $t \geq 5R$. Recall that

$$5R \leq t \leq 10R - 1.$$

We use a latin square of the form $I_R \times L_5$, where I_R is a symmetric latin square of order R. Note that we can split $I_R \times L_5$ into R 'diagonals' of 5×5 latin squares (each isomorphic to L_5), or more particularly, a centre diagonal plus $(R-1)/2$ pairs of symmetric diagonals, as shown in Figure 5.5.

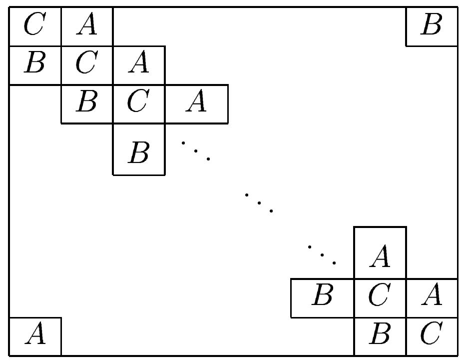


Figure 5.5

The subsquares marked C represent the centre diagonal; the subsquares marked A and B represent the innermost pair of diagonals.

For the centre diagonal we have three options for the R latin squares of order 5:

(i) Place R copies of $K(5,5,5)$ (or trade $25R$ 3-cycles with $15R$ 5-cycles).

(ii) Place R copies of $K(5,5,7)$ using the entries $5R+1, 5R+2$ from the end of each row and column.

(iii) Place R copies of $K(5,5,9)$ using the entries $5R+1, 5R+2, 5R+3, 5R+4$ from the end of each row and column.

We now "pair off" the remaining diagonals, starting from the innermost pair and working outwards. For each pair we have four options:

(i) Place $2R$ copies of $K(5,5,5)$.

(ii) Place $2R$ copies of $K(5,5,7)$ using four entries from the end of each row and column.

(iii) Place $2R$ copies of $K(5,5,9)$ using eight entries from the end of each row and column.

(iv) Label the diagonals A and B. Pair off each cell (i,j) in A with the cell (j,i) in B, which will contain the same entry. We then place trades of type $1B$ on each pair of cells, using two entries from the end of row i and two entries from the bottom of column i for each pair. Ten entries are used from the end of each row and column.

It is irrelevant which entries are used at the end of each row and column, provided the *same* entries are used by each pair of diagonals.

We now show that by choosing the right trades for each diagonal, we may decompose $K(5R, 5R, t)$ into 5-cycles for any odd t, $5R \leq t \leq 10R - 1$. First note that

$$0 \leq t - 5R \leq 5R - 1$$

and that $t - 5R$ is the number of entries at the end of each row and column (and of course is even). The main diagonal uses $0, 2$ or 4 entries from the end of each row and column, while each of the paired diagonals uses either $0, 4, 8$ or 10 entries from the end of each row or column.

So the total number of entries used at the ends of rows and columns must belong to the following set:

$$Q = \{0, 2, 4\} \oplus \frac{R-1}{2} * \{0, 4, 8, 10\}.$$

But

$$\{0, 2, 4, \ldots, 5R - 1\} \subseteq Q,$$

so our construction gives all the desired cases. ▮

THEOREM 5.19 *Let $K(10R, 10R, t)$ be a complete tripartite graph with $t \geq 10R$ and t even, that satisfies the necessary conditions given in Theorem 2.1. Then there exists a decomposition of $K(10R, 10R, t)$ into 5-cycles.*

Proof

¿From condition (iii) of Theorem 2.1, we have $t \leq 20R$. First consider the case $R = 1$. ¿From Lemma 4.15, the complete tripartite graphs $K(5, 5, 5)$, $K(5, 5, 7)$ and $K(5, 5, 9)$ all decompose into 5-cycles. So by Corollary 2.6, $K(10, 10, t)$ decomposes into 5-cycles for any even t between 10 and 18. Also, from Theorem 2.4, $K(2, 2, 4)$ decomposes into 5-cycles. Therefore using Corollary 2.8, $K(10, 10, 20)$ decomposes into 5-cycles.

Next consider the case $R > 1$. Now,

$$t \in \{10R, 10R + 2, \ldots, 20R\} \subseteq R * \{10, 12, \ldots, 20\}.$$

Therefore using Corollary 2.6, $K(10R, 10R, t)$ decomposes into copies of $K(10, 10, 10)$, $K(10, 10, 12)$, $K(10, 10, 14)$, $K(10, 10, 16)$, $K(10, 10, 18)$ and $K(10, 10, 20)$, which in turn decompose into 5-cycles. ▮

6 The case r and s divisible by 10

In this case we use a slightly different technique. We use a particular matrix representation M_1 as follows. Define

$$i \circ j = \begin{cases} i + j - 3 \pmod{t} & \text{if } i \text{ and } j \text{ are even;} \\ i + j - 1 \pmod{t} & \text{otherwise.} \end{cases}$$

Let L_1 be the $r \times s$ latin rectangle with (i,j) entry $i \circ j$, for $1 \leq i \leq r$, $1 \leq j \leq s$. Then M_1 arises from L_1 by adjoining to the ends of its rows and columns those entries from $\{1,2,\ldots,t\}$ which have not already occurred in each row and each column.

Now we construct some new trades, combining smaller trades of type 2. The trades in Figure 6.1 use only half of the entries within a subrectangle. As shown, these trades use all odd entries in a subrectangle; however, by a relabelling of entries and swapping rows and columns, the trades can instead use all even entries of a subrectangle instead.

$\mathbf{1}_1$		$\mathbf{3}_1$		$\mathbf{5}_1$		$\mathbf{7}_2$		$\mathbf{9}_2$	
	$\mathbf{1}_3$		$\mathbf{3}_3$		$\mathbf{5}_3$		$\mathbf{7}_4$		$\mathbf{9}_4$
$\mathbf{3}_1$		$\mathbf{5}_1$		$\mathbf{7}_2$		$\mathbf{9}_2$		$\mathbf{11}_2$	
	$\mathbf{3}_3$		$\mathbf{5}_3$		$\mathbf{7}_4$		$\mathbf{9}_4$		$\mathbf{11}_4$

$1,2,3,4 : 2B$

Trade type $3C$

$\mathbf{1}_1$		$\mathbf{3}_1$		$\mathbf{5}_1$		$\mathbf{7}_3$		$\mathbf{9}_3$	
	$\mathbf{1}_4$		$\mathbf{3}_4$		$\mathbf{5}_4$		$\mathbf{7}_6$		$\mathbf{9}_6$
$\mathbf{3}_1$		$\mathbf{5}_2$		$\mathbf{7}_2$		$\mathbf{9}_2$		$\mathbf{11}_3$	
	$\mathbf{3}_4$		$\mathbf{5}_5$		$\mathbf{7}_5$		$\mathbf{9}_5$		$\mathbf{11}_6$
$\mathbf{5}_1$		$\mathbf{7}_2$		$\mathbf{9}_2$		$\mathbf{11}_3$		$\mathbf{13}_3$	
	$\mathbf{5}_4$		$\mathbf{7}_5$		$\mathbf{9}_5$		$\mathbf{11}_6$		$\mathbf{13}_6$

$1,4 : 2F \quad 2,5 : 2B \quad 3,6 : 2A$

Trade type $3D$

Figure 6.1

Since r and s are both divisible by 10, our latin rectangle may be partitioned into 10×10 subsquares. In Lemma 6.20 we construct an array of pairs of integers that will be replaced by 10×10 subsquares in Theorem 6.21. Each pair of integers will indicate what type of trade is to be placed on each 10×10 subsquare.

LEMMA 6.20 *Let a, b, m and n be non-negative integers such that $m \leq n$, $0 < b \leq a \leq 10m-2$, $b \leq 10n$ and $mb + na \leq 10mn$.*

Then we can construct an $m \times n$ array with the ordered pair of integers (x_{ij}, y_{ij}) in each cell (i,j), $1 \leq i \leq m$ and $1 \leq j \leq n$, such that:

1. $x_{ij}, y_{ij} \in \{0,1,2,\ldots,10\}$;
2. $x_{ij} + y_{ij} \leq 10$;
3. $\sum_{i=1}^{m} x_{ij} = a$;
4. $\sum_{j=1}^{n} y_{ij} = b$;
5. $(x_{ij}, y_{ij}) \neq (0,9)$ *and*
6. $(x_{ij}, y_{ij}) \neq (9,0)$.

Proof

Let $a = km + u$ and $b = ln + v$, where k, l, u and v are non-negative integers with $k \leq 10$, $l \leq 10$, $u < m$ and $v < n$. From the inequality $mb + na \leq 10mn$ we may deduce that

$$k + l \quad \leq \quad 10 - (un + vm)/mn. \qquad (1)$$

First consider the case when $k + l = 10$. Here $u = v = 0$, and we set $x_{ij} = k$ and $y_{ij} = l$ for all i, j, $1 \leq i \leq m$ and $1 \leq j \leq n$. Each pair of integers (x_{ij}, y_{ij}) clearly satisfies our necessary conditions.

Next consider when $k + l = 9$. Let S be the following set of cells:

$$\{(i, j) \mid 1 \leq j \leq n, (j-1)u + 1 \leq i \leq ju\}$$

where i is calculated modulo m. If $(i, j) \in S$, we set x_{ij} to be $k+1$ and x_{ij} to be l. In every cell not belonging to S we set x_{ij} to be l.

The set S has nu entries spread as evenly as possible in the columns, as indicated in Figure 6.2.

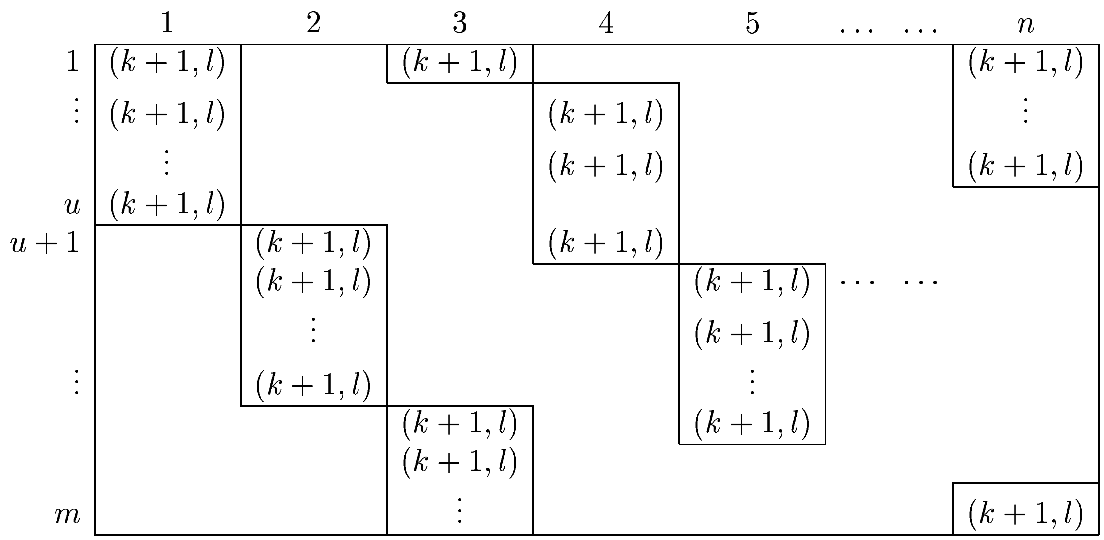


Figure 6.2

There are mn cells in our array altogether and S contains nu of them. ¿From (1) we have

$$mn - nu \geq vm.$$

Therefore the set of cells which do not belong to S intersects each row at least v times. So we may indeed set each y_{ij} in the complement of S to be either l or $l + 1$ so that $\sum_{i=1}^{m} y_{ij} = ln + v$ when $1 \leq j \leq n$. Conditions 1 to 4 are thus satisfied.

Next consider when $k+l \leq 8$. Then $(k+1)+(l+1) \leq 10$, so we may set $x_{ij} = k+1$ and $y_{ij} = l + 1$ in a cell without contradiciting condition 2. In each column we set $x_{ij} = k+1$ in u cells and $x_{ij} = k$ in the remaining cells. In each row we set $y_{ij} = l+1$ in v cells and $y_{ij} = l$ in the remaining cells.

To ensure conditions 5 and 6 hold, first recall that $b \leq a \leq 10m - 2$. Then clearly $l \leq k$, so certainly we cannot have $x_{ij} = 0$ and $y_{ij} = 9$. Suppose there exists a cell

(i,j) with $x_{ij} = 9$ and $y_{ij} = 0$. Then since $a \leq 10m-2$ there exists another cell (i',j), $i' \neq i$ such that $x_{i'j} < 10$. If $x_{i'j} = 8$ and $y_{i'j} = 0$ we increase $x_{i'j}$ by 2 and decrease x_{ij} by 2. Otherwise increase x_{ij} by 1 and decrease $x_{i'j}$ by 1. Note that conditions 1 to 4 still hold. Repeat this until no such cells exist! ∎

The following theorem excludes the case when $s = t$. The case when $s = t$ and both s and t are divisible by 10 was dealt with in Theorem 5.16.

THEOREM 6.21 *Let $K(r,s,t)$ be a complete tripartite graph with $r \leq s < t$ such that r, s and t are all even, and $r \equiv s$ (modulo 10). Then if the necessary conditions of Theorem 2.1 are satisfied, $K(r,s,t)$ can be decomposed into 5-cycles.*

Proof

In this theorem we shall always use the matrix representation M_1.

Let P_{ij} be the 10×10 subsquare of the form

$$\{(i,j) \mid 10I-9 \leq i \leq 10I, 10J-9 \leq j \leq 10J\},$$

for $1 \leq I \leq r/10$ and $1 \leq J \leq s/10$. Note that the subsquares of the form P_{ij} form a partition of our latin rectangle L.

We define a *diagonal* of a subsquare P_{ij} to be a set of ten entries such that each row and column contains an entry and the set may be partitioned into 5 pairs of the same entry. Each subsquare P_{ij} may be partitioned into 10 diagonals. Each diagonal may be used by trades of type $1A$ (or the transpose). In this way we may use $2k$ entries from the bottom of each of the ten columns intersecting the subsquare and $20-2k$ entries from the end of each of the ten rows intersecting the subsquare, where k is any integer from 0 to 10.

We may also wish to use fewer than 10 diagonals with trades of type 1. The following trades demonstrate how this may be done. Each trade uses a number of diagonals from one of our 10×10 subsquares. For example, the trade in Figure 6.3 uses two diagonals with trades of type 2.

1	2	3	$\mathbf{4_1}$	5	$\mathbf{6_1}$	7	8	9	10
2	1	$\mathbf{4_2}$	3	$\mathbf{6_2}$	5	8	7	10	9
3	$\mathbf{4_1}$	5	$\mathbf{6_1}$	7	8	9	10	11	12
$\mathbf{4_2}$	3	$\mathbf{6_2}$	5	8	7	10	9	12	11
5	$\mathbf{6_1}$	7	8	9	10	11	12	13	$\mathbf{14_3}$
$\mathbf{6_2}$	5	8	7	10	9	12	11	$\mathbf{14_4}$	13
7	8	9	10	11	12	13	$\mathbf{14_3}$	15	$\mathbf{16_3}$
8	7	10	9	12	11	$\mathbf{14_4}$	13	$\mathbf{16_4}$	15
9	10	11	12	13	$\mathbf{14_3}$	15	$\mathbf{16_3}$	17	18
10	9	12	11	$\mathbf{14_4}$	13	$\mathbf{16_4}$	15	18	17

Subscripts : 1, 2, 3, 4 : 2J

Figure 6.3. Trade type $3E$

For each trade, the remaining diagonals may be used by trades of type $1A$ (or the transpose).

The trade in Figure 6.4 uses three diagonals with trades of type 2.

1	2	3	$\mathbf{4_6}$	5	$\mathbf{6_6}$	7	8	9	$\mathbf{10_1}$
2	1	$\mathbf{4_5}$	3	$\mathbf{6_4}$	5	8	7	$\mathbf{10_4}$	9
3	$\mathbf{4_6}$	5	$\mathbf{6_6}$	7	8	9	$\mathbf{10_3}$	11	12
$\mathbf{4_5}$	3	$\mathbf{6_5}$	5	8	7	$\mathbf{10_5}$	9	12	11
5	$\mathbf{6_6}$	7	8	9	$\mathbf{10_1}$	11	12	13	$\mathbf{14_1}$
$\mathbf{6_4}$	5	8	7	$\mathbf{10_4}$	9	12	11	$\mathbf{14_2}$	13
7	8	9	$\mathbf{10_3}$	11	12	13	$\mathbf{14_3}$	15	$\mathbf{16_3}$
8	7	$\mathbf{10_5}$	9	12	11	$\mathbf{14_2}$	13	$\mathbf{16_2}$	15
9	$\mathbf{10_1}$	11	12	13	$\mathbf{14_1}$	15	$\mathbf{16_3}$	17	18
$\mathbf{10_4}$	9	12	11	$\mathbf{14_2}$	13	$\mathbf{16_2}$	15	18	17

Subscripts : $1, 2, 4, 6$: 2J $3, 5$: 2F

Figure 6.4. Trade type $3F$

To use four diagonals, we use the trade in Figure 6.5.

1	$\mathbf{2_1}$	3	$\mathbf{4_1}$	5	$\mathbf{6_4}$	7	$\mathbf{8_1}$	9	10
$\mathbf{2_2}$	1	$\mathbf{4_2}$	3	$\mathbf{6_3}$	5	$\mathbf{8_2}$	7	10	9
3	$\mathbf{4_1}$	5	$\mathbf{6_4}$	7	$\mathbf{8_4}$	9	10	11	$\mathbf{12_5}$
$\mathbf{4_2}$	3	$\mathbf{6_3}$	5	$\mathbf{8_3}$	7	10	9	$\mathbf{12_6}$	11
5	$\mathbf{6_4}$	7	$\mathbf{8_4}$	9	10	11	$\mathbf{12_8}$	13	$\mathbf{14_7}$
$\mathbf{6_3}$	5	$\mathbf{8_3}$	7	10	9	$\mathbf{12_7}$	11	$\mathbf{14_8}$	13
7	$\mathbf{8_1}$	9	10	11	$\mathbf{12_8}$	13	$\mathbf{14_7}$	15	$\mathbf{16_5}$
$\mathbf{8_2}$	7	10	9	$\mathbf{12_7}$	11	$\mathbf{14_8}$	13	$\mathbf{16_6}$	15
9	10	11	$\mathbf{12_5}$	13	$\mathbf{14_7}$	15	$\mathbf{16_5}$	17	$\mathbf{18_5}$
10	9	$\mathbf{12_6}$	11	$\mathbf{14_8}$	13	$\mathbf{16_6}$	15	$\mathbf{18_6}$	17

Subscripts : $1, 2, 5, 6$: 2F $3, 4, 7, 8$: 2J

Figure 6.5. Trade type $3G$

The trade in Figure 6.6 uses five diagonals.

$\mathbf{1_1}$	2	$\mathbf{3_1}$	4	$\mathbf{5_1}$	6	$\mathbf{7_1}$	8	$\mathbf{9_1}$	10
2	$\mathbf{1_1}$	4	$\mathbf{3_1}$	6	$\mathbf{5_1}$	8	$\mathbf{7_1}$	10	$\mathbf{9_1}$
$\mathbf{3_1}$	4	$\mathbf{5_1}$	6	$\mathbf{7_1}$	8	$\mathbf{9_1}$	10	$\mathbf{11_1}$	12
4	$\mathbf{3_1}$	6	$\mathbf{5_1}$	8	$\mathbf{7_1}$	10	$\mathbf{9_1}$	12	$\mathbf{11_1}$
$\mathbf{5_1}$	6	$\mathbf{7_1}$	8	$\mathbf{9_1}$	10	$\mathbf{11_1}$	12	$\mathbf{13_1}$	14
6	$\mathbf{5_1}$	8	$\mathbf{7_1}$	10	$\mathbf{9_1}$	12	$\mathbf{11_1}$	14	$\mathbf{13_1}$
$\mathbf{7_2}$	8	$\mathbf{9_2}$	10	$\mathbf{11_2}$	12	$\mathbf{13_2}$	14	$\mathbf{15_2}$	16
8	$\mathbf{7_2}$	10	$\mathbf{9_2}$	12	$\mathbf{11_2}$	14	$\mathbf{13_2}$	16	$\mathbf{15_2}$
$\mathbf{9_2}$	10	$\mathbf{11_2}$	12	$\mathbf{13_2}$	14	$\mathbf{15_2}$	16	$\mathbf{17_2}$	18
10	$\mathbf{9_2}$	12	$\mathbf{11_2}$	14	$\mathbf{13_2}$	16	$\mathbf{15_2}$	18	$\mathbf{17_2}$

Subscripts : 1 : 3D 2 : 3C

Figure 6.6. Trade type $3H$

To use six diagonals within a 10×10 subsquare, combine a trade of type $3E$ with one of type $3G$, relabelling the entries in $3E$ so that odd rather than even entries are used. To use seven diagonals we combine $3E$ with $3H$, to use eight we combine $3F$ with $3H$, to use nine we combine $3G$ with $3H$ and finally to use all ten diagonals we use two trades of type $3H$, with one trade using odd entries and the other using even entries.

It is not possible to use only one diagonal with trades of type 2. However, with the trade in Figure 6.7 we may use $2k$ entries from the end of each row and $2l$ entries from the bottom of each column, for integers $k > 0$, $l > 0$ such that $2k + 2l = 18$.

$\mathbf{1_1}$	2	$\mathbf{3_1}$	4	5	6_3	7	8_4	9	10_5
2	$\mathbf{1_2}$	4	$\mathbf{3_2}$	6_3	5	8_4	7	10_5	9
$\mathbf{3_1}$	4	$\mathbf{5_1}$	6	7	8_4	9	10_5	11	12_3
4	$\mathbf{3_2}$	6	$\mathbf{5_2}$	8_4	7	10_5	9	12_3	11
$\mathbf{5_1}$	6	7	8	9	10_5	11	12_3	*13*	14_4
6	$\mathbf{5_2}$	8	7	10_5	9	12_3	11	14_4	*13*
7	8_4	9_3	10_5	*11*	12	13	14	15	16
8_4	*7*	10_5	9_3	12	*11*	14	13	16	15
9_3	10_5	*11*	12_4	13	14	*15*	16	17	18
10_5	9_3	12_4	*11*	14	13	16	*15*	18	17

Subscripts: $1, 2 : 2B$

Trade type 3I

Figure 6.7

Entries in italics are used by trades of type $1A$ (in transpose). The entries with subscript 3 form a diagonal; similarly entries with subscripts 4 and 5. Each diagonal is used by trades of type $1A$ (or its transpose). The remaining entries are used by trades of type $1A$, using ten entries from the end of each row. In this way we may use either 10, 12, 14 or 16 entries from the end of each row and 8, 6, 4 or 2 entries from the bottom of each column. By taking the transpose of the trade $3G$ we may use any number of trades from the ends of rows or columns except for 0 or 18.

With the help of Lemma 6.20 we can now indicate when and where each of the above trades are used in our latin rectangle L_1. Let $r = 10m$, $s = 10n$, $a = (t-r)/2$ and $b = (t-s)/2$. ¿From necessary conditions $t \leq 4rs/(r+s)$, $t \leq 3r$ and $t \leq 2s$ (see Theorem 2.1 and Corollary 2.2) we see that $mb + an \leq 10mn$, $a \leq 10m$ and $b \leq 10n$, respectively, all hold. Since $r \leq s$ we have $(t-s)/2 \leq (t-r)/2$, so $b \leq a$. If $a = 10m$, then $(t-r)/2 = r$, but this implies that $s = t$, a case not covered by this theorem. If $a = 10m - 1$, $(t-r)/2 = r - 1$ so that $t = 3r - 2$. Since $s \leq t$ and s is divisible by 10, we have that $s \leq 3r - 10$. But from the inequality $t \leq 4rs/(r+s)$ we have $s \geq r(3r-2)/(r+2)$. Thus $(3r-10)(r+2) \geq r(3r-2)$, which implies that r is negative, so we must have $a \leq 10m - 2$.

Therefore there exists an array of cells, each with a pair of entries (x_{ij}, y_{ij}) that satisfy the properties given in Lemma 6.20. We replace each cell (x_{ij}, y_{ij}) with a 10×10 subsquare P_{ij}. We apply one of the previous trades to the entries of P_{ij}, using $2x_{ij}$ entries from the bottoms of columns and $2y_{ij}$ entries from the ends of the rows.

Thus every entry in our matrix decomposition is used in a trade, and our decomposition is complete. ∎

7 Concluding remarks

We have now shown the following.

THEOREM 7.22 *The complete tripartite graph $K(r,s,t)$ (with $r \leq s \leq t$) decomposes into 5-cycles only if r, s and t are either all odd or all even, 5 divides $rs+st+rt$ and $t \leq 4rs/(r+s)$. These necessary conditions are sufficient in the case when two partite sets have equal size or in the case when r and s are divisible by 10.*

Thus to complete the sufficiency of the conditions for a decomposition of a complete tripartite graph into 5-cycles, we must solve the cases when r, s and t are all different, and r and s are not necessarily divisible by 10. The techniques used in Sections 5 and 6 should prove very useful in solving the remaining cases.

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