Balanced critical sets in Latin squares

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Abstract

In this note we first introduce balanced critical sets and near balanced critical sets in Latin squares. Then we prove that there exist balanced critical sets in the back circulant Latin squares of order $3n$ for n even. Using this result we decompose the back circulant Latin squares of order $3n$, n even, into three isotopic and disjoint balanced critical sets each of size $3n^2$. We also find near balanced critical sets in the back circulant Latin squares of order $3n$ for n odd. Finally, we examine representatives of each main class of Latin squares of order up to six in order to determine which main classes contain balanced critical sets.

1 Introduction

A *Latin square* L of order n is an $n \times n$ array of integers chosen from a set N, of size n, such that each element of N occurs exactly once in each row and exactly once in each column. In this paper $N = \{0, 1, \ldots, n-1\}$, and the rows and columns of L are also indexed by N. We may also represent a Latin square as a set of ordered triples

$$\{(i, j; k) \mid i, j \in N, \text{ the cell } (i, j) \text{ contains entry } k\}.$$

The *back circulant* Latin square, denoted as BC_n, may be defined as follows:

$$BC_n = \{(i, j; i + j \pmod{n}) \mid i, j \in N\}.$$

A *partial Latin square* P of order n is an $n \times n$ array of integers chosen from a set N, of size n, such that each element occurs *at most* once in each row or column. Thus a partial Latin square may contain empty cells. A partial Latin square P of order n is said to be *uniquely completable* if it may be embedded in exactly one Latin square L of order n.

A partial Latin square P is said to be a *critical set* of a Latin square L if:

1. P is uniquely completable to L;

2. no proper subset of P satisfies 1.

It turns out that critical sets may also be defined in terms of *Latin trades*. We define the *shape* of a partial Latin square P to be the set $\{(i,j) \mid (i,j;k) \in P\}$. That is, the shape of P is the set of cells in P that are not empty. The *size* of P is the number of non-empty cells in P. Let I and I' be partial Latin squares with the same shape and size. We say that I and I' are *mutually balanced* if every row (column) of I contains the same entries (though perhaps in a different order) as the corresponding row (column) of I'. If I and I' have the same shape, size, are mutually balanced and $I \cap I' = \emptyset$ then I is said to be a *Latin trade* (or *Latin interchange*) and I' is its *disjoint mate*.

Let L be a Latin square and I a Latin trade contained in L. We may then replace I with its disjoint mate I' to form a new Latin square L'. It follows that a critical set C of L must intersect each Latin trade within L non-trivially. In fact, a critical set C of L can also be defined by the following conditions:

1. If I is Latin trade within L, then $|I \cap C| \geq 1$;

2. for each $(i,j;k) \in C$ there exists a Latin trade I within L such that $I \cap C = \{(i,j;k)\}$.

If P is a partial Latin square in L and for a particular element $(i,j;k) \in P$, there exists a Latin trade I in L such that $I \cap P = \{(i,j;k)\}$, we say that the element $(i,j;k)$ is *necessary* (in the Latin square L). So to prove a partial Latin square P is a critical set of a Latin square L it suffices to show that P completes uniquely to L and that each $(i,j;k) \in P$ is necessary in L. It is precisely this argument that will be used in this paper to prove the existence of a new class of critical sets. Hence the construction of Latin trades is integral to our proof.

Critical sets in Latin squares have been studied in the past (see for example [13, 8]). Various properties for critical sets such as critical sets of smallest sizes ([3, 10]), critical sets of largest sizes ([1]), near-strong critical sets ([5]) and totally weak critical sets ([4]) have also been studied. Adams, Bean and Khodkar found disjoint critical sets in Latin squares ([2]). Donovan and Khodkar [12] defined and studied uniform critical sets in Latin squares. In this note we introduce and study balanced critical sets and near balanced critical sets in Latin squares.

A critical set is called *(k-)balanced* if

1. each row contains precisely k entries;
2. each column contains precisely k entries;
3. each entry occurs in precisely k cells.

Hence the size of a k-balanced critical set of order n must be kn. A critical set is called *near (k-)balanced* if

1. there are $n-1$ rows, each containing k entries and one row that contains $k-1$ entries;
2. there are $n-1$ columns, each containing k entries and one column that contains precisely $k-1$ entries;
3. there are $n-1$ entries, each occurring in precisely k cells and one entry that occurs in precisely $k-1$ cells.

Hence the size of a near k-balanced critical set of order n must be $kn-1$.

Note that if L and L' are Latin squares of order n that are either isotopic or in the same main class then there is a one-to-one correspondence between (near) k-balanced critical sets in L and (near) k-balanced critical sets in L'. (see [11] for a definition of isotopic and a list of main classes of Latin squares of small order.)

We prove that there exist n-balanced critical sets in the back circulant Latin squares of order $3n$ for n even. In [2] it was proved that the back circulant Latin square of order n, $n > 1$, can be decomposed into four disjoint critical sets. We decompose the back circulant Latin squares of order $3n$, n even, into three isotopic and disjoint balanced critical sets each of size $3n^2$. We also find near n-balanced critical sets in the back circulant Latin squares of order $3n$ for n odd. Finally, we examine representatives of each main class of Latin square (see [6]) of order up to six in order to determine which main classes contain balanced critical sets.

Next we introduce some partial Latin squares in the back circulant Latin square that have already been shown to be critical sets. Let $\varepsilon_{n,m}$ be the following partial Latin square in BC_n, where m is an integer less than n:

$$\begin{aligned}\varepsilon_{n,m} = \quad & \{(i,j;i+j \pmod{n}) \mid i+j < m\} \\ & \cup\{(i,j;i+j-n)\} \mid i+j \geq n+m\}.\end{aligned}$$

THEOREM 1.1 ([8], [7] and [9]) *The partial Latin square $\varepsilon_{n,m}$, defined above, is a critical set in BC_n, for all values of m less than n.*

The following definition gives the partial Latin squares that will be of chief consideration in this paper.

DEFINITION 1.2 *Let $n \geq 1$ be an integer. We define C_{3n} to be the following partial Latin square in the back circulant Latin square BC_{3n}:*

$$C_{3n} = \{(i,j;i+j) \mid 0 \leq i,j \leq n-1\} \cup$$
$$\{(i,j;i+j(\text{mod } 3n) \mid n \leq i,j \leq 2n-1\} \cup$$
$$\{(i,j;i+j(\text{mod } 3n) \mid 2n \leq i,j \leq 3n-1\}.$$

We will show that if $n \geq 2$ and n is even, C_{3n} is a critical set in BC_{3n}. Also we will prove that if $n \geq 1$ and n is odd, $C_{3n} \setminus \{((3n-1)/2, (3n-1)/2; 3n-1)\}$ is a critical set in BC_{3n}.

EXAMPLE 1.3 *Figure 1 shows the partial Latin square $C_9 \setminus \{(4,4;8)\}$.*

0	1	2						
1	2	3						
2	3	4						
			6	7	8			
			7		0			
			8	0	1			
						3	4	5
						4	5	6
						5	6	7

Figure 1

2 New Latin trades

In this section we construct Latin trades that are used later to show that elements of C_{3n} are necessary.

THEOREM 2.4 *Let $x, y \geq 1$. Consider the subrectangle in the Latin square BC_{x+y} cornered by the following elements:*

$$(0,0;0), (0,y;y), (x,0;x) \text{ and } (x,y;0).$$

Then there exists a Latin trade, denoted by $I_{x,y}$, with the following properties:

1. *$I_{x,y}$ is contained within the above subrectangle.*
2. *$I_{x,y}$ includes the above four elements.*
3. *The disjoint mate of $I_{x,y}$, denoted by $I'_{x,y}$, includes the elements $(0,y;0)$ and $(x,0;0)$.*

Proof First assume that $x \leq y$. Let $y = ax + b$, where a is an integer greater than or equal to 1 and $0 \leq b < x$. Our proof is by induction on b. If $b = 0$ our Latin trade is as follows:

$$I_{x,y} = \{(0, ix; ix), (x, ix; (i+1)x \pmod{x+y}) \mid 0 \leq i \leq a\},$$

with a disjoint mate obtained by swapping the entries in each column:

$$I'_{x,y} = \{(0, ix; (i+1)x \pmod{x+y}), (x, ix; ix) \mid 0 \leq i \leq a\}.$$

Otherwise $b \geq 1$. Assume that a Latin trade $I_{\alpha,\beta}$ satisfying conditions 1, 2 and 3 of this lemma exists for any integers α and β such that $\alpha \leq \beta$ and $\beta \pmod{\alpha} < b$. Let a' and b' be integers such that $x = a'b + b'$, where $a' \geq 0$ and $0 \leq b' < b$. Since $b' < b < x$, by our inductive assumption we know that the Latin trade $I_{b,x}$ exists.

We construct our Latin trade $I_{x,y}$ in this case as follows. Remove the element $(b, x; 0)$ from $I_{b,x}$, then take its transpose. Add ax to the entries of the resultant partial Latin square and place this in columns ax through to $ax + b$ of $I_{x,y}$. The remaining elements are placed as before in rows 0 and x, and columns 0, x, $2x$ through to $(a-1)x$. Finally we add the element $(x, y; 0)$. More formally:

$$\begin{aligned} I_{x,y} = & \{(0, ix; ix), (x, ix; (i+1)x) \mid 0 \leq i \leq a-1\} \\ & \cup\{(j, i+ax; k+ax) \mid (i,j;k) \in I_{b,x} \setminus \{(b, x; 0)\}\} \cup \{(x, y; 0)\}, \end{aligned}$$

with disjoint mate:

$$\begin{aligned} I'_{x,y} = & \{(0, ix; (i+1)x), (x, ix; ix) \mid 0 \leq i \leq a-1\} \\ & \cup\{(j, i+ax; k+ax) \mid (i,j;k) \in I'_{b,x} \setminus \{(b, 0; 0)\}\} \cup \{(0, y; 0)\}. \end{aligned}$$

The inductive step just given is clarified for the reader in Figure 2.

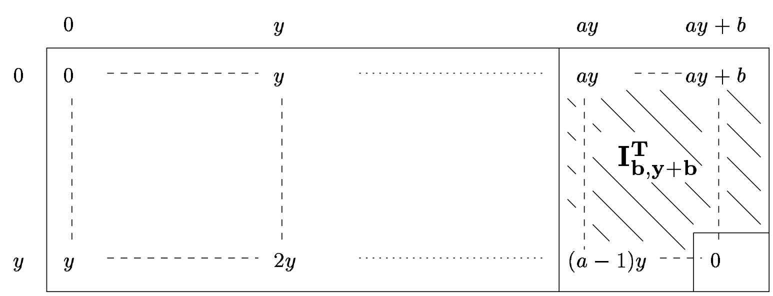

Figure 2: The inductive step in Theorem 2.4

Finally if $x > y$, we let $I_{x,y} = (I_{y,x})^T$. ∎

EXAMPLE 2.5 *Figure 3 shows the Latin trade $I_{5,8}$, together with its disjoint mate $I'_{5,8}$, constructed as in Theorem 2.4.*

0					5			8
					8		10	11
							11	12
5					10		12	0

$I_{5,8}$

5					8			0
					10		11	0
							12	11
0					5		10	12

$I'_{5,8}$

Figure 3

3 The necessity of elements

Here we show that every element of C_{3n} (if n is even) and every element of $C_{3n} \setminus \{((3n-1)/2, (3n-1)/2; 3n-1)\}$ (if n is odd) is necessary in the Latin square BC_{3n}. We do this by constructing Latin trades that only intersect one element of the partial Latin square under consideration.

Because of the cyclic nature of the back circulant Latin square, any partial Latin square that it contains may be "shifted", as described in the following definition.

DEFINITION 3.6 *Let P be a partial Latin square in the back circulant latin square BC_m, for some integer $m \geq 2$. Then we define $P \oplus (i,j)$ to be the following partial Latin square, also contained in BC_m:*

$$P \oplus (i,j) = \{(\alpha + i, \beta + j; \gamma + i + j) \mid (\alpha, \beta; \gamma) \in P\}.$$

(All rows, columns and entries are calculated modulo m.)

Using this definition we can define C_{3n} as follows. Let Q_n be the partial Latin square in BC_{3n} formed by the intersection of rows n through to $2n-1$ with columns n through to $2n-1$:

$$Q_n = \{(i,j; i+j) \mid n \leq i, j \leq 2n-1\}.$$

Then

$$C_{3n} = Q_n \cup (Q_n \oplus (n,n)) \cup (Q_n \oplus (2n,2n)).$$

Also, observe that

$$C_{3n} = C_{3n} \oplus (n,n) = C_{3n} \oplus (2n,2n).$$

Now, if I is a Latin trade, it is also a partial Latin square, so we can apply Definition 3.6 to I. It follows that if n is even, we only need to show that the elements of Q_n are necessary for a unique completion. For if I is a Latin trade and $I \cap C_{3n} = \{(i, j; k)\}$, then

$$\begin{aligned} (I \oplus (n, n)) \cap C_{3n} &= \{(i + n, j + n; k + 2n)\} \text{ and} \\ (I \oplus (2n, 2n)) \cap C_{3n} &= \{(i + 2n, j + 2n; k + n)\} \end{aligned}$$

are also Latin trades (with rows, columns and entries calculated modulo $3n$). Thus if $(i, j; k) \in Q_n$ is necessary, the elements $(i + n, j + n; k + n)$ and $(i + 2n, j + 2n; k + 2n)$ are also necessary.

We can make more simplifications. The Latin square BC_{3n} and the partial Latin square C_{3n} are both symmetric, and the transpose of a Latin trade is also a Latin trade. Thus we only need look at the elements $(i, j; k) \in Q_n$ for which $j \geq i$.

Finally observe that C_{3n} and BC_{3n} have a symmetric pattern about the back diagonal (the diagonal containing the entries $3n - 1$). We can "flip" C_{3n} on the axis of this diagonal to obtain a partial Latin square isotopic to C_{3n}. In other words,

$$C_{3n} = \{(3n - 1 - i, 3n - 1 - j; 3n - 2 - k \pmod{3n}) \mid (i, j; k) \in C_{3n}\}.$$

Thus, we may also assume for an element $(i, j; k) \in Q_n$ that $i + j \leq 3n - 1$.

If n is odd, we remove element $((3n-1)/2, (3n-1)/2; 3n-1)$ from C_{3n}. To show that the elements $((n-1)/2, (n-1)/2; n-1)$ and $((5n-1)/2, (5n-1)/2; 2n-1)$ are necessary we construct Latin trades that use the removed element. But for all other elements in $C_{3n} \setminus \{((3n-1)/2, (3n-1)/2; 3n-1)\}$, the Latin trades to be constructed do not involve the missing element. Thus with the exception of two special cases we apply all of the above simplifications when n is odd.

THEOREM 3.7 *Let $n \geq 1$ be an integer. If n is even, for each $(i, j; k) \in C_{3n}$, there exists a Latin trade I in the back circulant Latin square BC_{3n} such that $I \cap C_{3n} = \{(i, j; k)\}$. Furthermore if n is odd, then for each $(i, j; k) \in C_{3n} \setminus \{((3n - 1)/2, (3n - 1)/2; 3n - 1)\}$, there exists a Latin trade $I \in BC_{3n}$ such that*

$$I \cap (C_{3n} \setminus \{((3n - 1)/2, (3n - 1)/2; 3n - 1)\}) = \{(i, j; k)\}.$$

Proof Consider an element $(i, j; k)$ that we wish to show is necessary in the Latin square BC_{3n}. From previous observations (and with two exceptions if n is odd), we may assume that $n \leq i \leq 2n - 1$, $n \leq j \leq 2n - 1$, $i \leq j$ and $i + j \leq 3n - 1$.

We split our proof into five cases altogether. If $i = j$ there are two cases: one for even n, the other for odd n. Otherwise $i \neq j$ and there are three cases: $j > 3n/2$, $j < 3n/2$ and $j = 3n/2$.

Case 1 $i = j$, n even. Since $i + j \leq 3n - 1$, we know that $i \leq 3n/2 - 1$.

First consider the element $(2, 2; 4) \in C_6$. A Latin trade I_1, together with its disjoint mate I_1', is given in the Figure 4. The elements of I_1 and I_1' are shown in bold.

0	1	**2**	3	**4**	**5**
1	2	3	4	**5**	**0**
2	**3**	**4**	5	**0**	**1**
3	**4**	5	0	**1**	**2**
4	5	0	1	2	3
5	0	1	2	3	4

I_1

0	1	**4**	3	**5**	**2**
1	2	3	4	**0**	**5**
3	**4**	**2**	5	**1**	**0**
2	**3**	5	0	**4**	**1**
4	5	0	1	2	3
5	0	1	2	3	4

I_1'

Figure 4

Note that $I \cap C_6 = \{(2, 2; 4)\}$. Thus $(2, 2; 4)$ is necessary in C_6.

Next consider the element $(n, n; 2n) \in C_{3n}$. We can obtain a Latin trade I_2 that intersects C_{3n} only at the element $(n, n; 2n)$ by taking the above Latin trade in BC_6 and multiplying every row, column and entry by $n/2$. Finally the Latin trade $I_2 \oplus (i - n, i - n)$ intersects C_{3n} only in the element $(i, i; 2i)$.

Case 2 $i = j$, n odd. Let $m = (n - 1)/2$. First consider two special elements: $(m, m; n - 1)$ and $(2n + m, 2n + m; 2n - 1)$. For these elements we construct Latin trades that use the "missing" element $(n + m, n + m; 3n - 1)$. The Latin trades are:

$$\{(m, m; n - 1), (m, n + m; 2n - 1), (2n + m, m; 3n - 1),\\ (n + m, n + m; 3n - 1), (n + m, m; 2n - 1), (2n + m, n + m; n - 1)\}$$

and

$$\{(m, 2n + m; 3n - 1), (m, n + m; 2n - 1), (2n + m, 2n + m; 3n - 1),\\ (n + m, n + m; 3n - 1), (n + m, 2n + m; n - 1), (2n + m, n + m; n - 1)\}.$$

All other Latin trades in this theorem *do not use* the element $(n + m, n + m; 3n - 1)$.

So we can now assume that $(i, i; 2i) \in Q_n$, and $i < n + m$. This is now similar to Case 1. First we construct a Latin trade for the element $(n, n; 2n)$:

$$\{(0, n; n), (0, 2n; 2n), (0, 2n + m; 2n + m), (m, 2n; 2n + m)\} \cup\\ \{(\alpha, 3n - 1 - \alpha; 3n - 1), (\alpha + 1, 3n - 1 - \alpha; 0) \mid m \leq \alpha \leq n - 1\} \cup\\ \{(n, 0; n), (n, n; 2n), (n + m + 1, m; 2n), (n + m + 1, 2n + m; n)\} \cup\\ \{(\alpha, n + m - \alpha; n + m) \mid n \leq \alpha \leq n + m\} \cup\\ \{(\alpha + 1, n + m - \alpha; n + m + 1) \mid n \leq \alpha \leq n + m\} \cup\\ \{(\alpha, 3n + m - \alpha; m), (\alpha + 1, 3n + m - \alpha; m + 1) \mid n \leq \alpha \leq n + m\},$$

with disjoint mate:

$$\begin{aligned}
&\{(0,n;2n),(0,2n;2n+m),(0,2n+m;n),(m,2n;3n-1)\}\cup\\
&\{(m,2n+m;2n+m)\}\cup\\
&\{(\alpha,3n-1-\alpha;0),(\alpha,3n-\alpha;3n-1)\mid m+1\le\alpha\le n-1\}\cup\\
&\{(n,0;n+m),(n,m;2n),(n,n;n),(n,2n;m),(n,2n+m;0)\}\cup\\
&\{(\alpha,n+m-\alpha;n+m+1)\mid n+1\le\alpha\le n+m\}\cup\\
&\{(\alpha,n+m+1-\alpha;n+m)\mid n+1\le\alpha\le n+m\}\cup\\
&\{(\alpha,3n+m-\alpha;m+1),(\alpha,3n+m+1-\alpha;m)\mid n+1\le\alpha\le n+m\}\cup\\
&\{(m+n+1,0;n),(m+n+1,m;m+n+1),(m+n+1,2n;2n)\}\cup\\
&\{(m+n+1,2n+m;m+1)\}.
\end{aligned}$$

For other elements in this case use the Latin trade $I\oplus(i-n,i-n)$. An example of this case is given in Example 3.8.

Case 3 $i<j$ and $j<3n/2$. Here we construct a Latin trade cornered by the elements $(i,0;i)$, $(i,j;i+j)$, $(i-j+3n,0;i-j+3n)$ and $(i-j+3n,j;i)$. First construct the Latin trade $I_{3n-2j,j}$ from Lemma 2.4. Then remove from $I_{3n-2j,j}$ the element $(3n-2j,j;0)$. Without this corner element, the Latin trade $I_{3n-2j,j}$ may lie in the back circulant square BC_{3n}. So we can "shift" $I_{3n-2j,j}\setminus\{(3n-2j,j;0)\}$ by $i+j$ rows to obtain $(I_{3n-2j,j}\setminus\{(3n-2j,j;0)\})\oplus(i+j,0)$, which lies between rows $i+j$ and $i-j+3n$. Finally we add the elements $(i,0;i)$, $(i,j;i+j)$ and $(i-j+3n,j;i)$ and what we have is a Latin trade that intersects C_{3n} only in the element $\{i,j;i+j\}$. The disjoint mate is as follows:

$$\{(i,0;i+j),(i,j;i),(i-j+3n,0;i)\}\cup((I'_{3n-2j,j}\setminus\{(3n-2j,0;0)\})\oplus(i+j,0)),$$

where $I'_{3n-2j,j}$ is the disjoint mate of $I_{3n-2j,j}$.

An example of this case is given in Example 3.9.

Case 4 $i<j$ and $j=3n/2$. Here n must be even, and we can use an intercalate (a Latin trade of size 4). This is given by

$$\{(i,0;i),(i,3n/2;3n/2+i),(3n/2+i,0;3n/2+i),(3n/2+i,3n/2;i)\}.$$

Case 5 $i<j$ and $j>3n/2$. This case works similarly to Case 3. We construct a Latin trade cornered by the same four elements $(i,0;i)$, $(i,j;i+j)$, $(i-j+3n,0;i-j+3n)$ and $(i-j+3n,j;i)$. This time we need the Latin trade $I_{3n-j,2j-3n}$ from Lemma 2.4. Then remove from $I_{3n-j,2j-3n}$ the element $(3n-j,2j-3n;0)$. Without this corner element, the Latin trade $I_{3n-j,2j-3n}$ may lie in the back circulant square BC_{3n}. So we can "shift" $I_{3n-j,2j-3n}\setminus\{(3n-j,2j-3n;0)\}$ by i rows to obtain $(I_{3n-j,2j-3n}\setminus\{(3n-j,2j-3n;0)\})\oplus(i,0)$, which lies between rows i and $i-j+3n$ and columns 0 and $2j-3n$. Now $i\ge n$ and $j\le 2n-1$, so what we have so far does not intersect C_{3n}. Finally we add the elements

$(i,j;i+j)$, $(i-j+3n,2j-3n;i+j)$ and $(i-j+3n,j;i)$ and what we have is a Latin trade that intersects C_{3n} only in the element $\{i,j;i+j\}$. The disjoint mate is as follows:

$$\{(i,2j-3n;i+j),(i,j;i),(i-j+3n,j;i+j)\}\cup$$
$$(I'_{3n-j,2j-3n}\setminus\{(0,2j-3n;0)\})\oplus(i,0),$$

where $I'_{3n-j,2j-3n}$ is the disjoint mate of $I_{3n-j,2j-3n}$. ∎

EXAMPLE 3.8 *Figure* 5 *demonstrates that the element* $(6,6;12)$ *is necessary in the partial Latin square* $C_{15}\setminus\{(7,7;14)\}$ *in* BC_{15}. *This is an example of Case* 2 *in Theorem* 3.7. *We show only the first ten rows of the partial Latin square. The elements of the Latin trade are shown in bold.*

0	1	2	3	4										
1	2	3	4	5		**7**					**12**		**14**	
2	3	4	5	6										
3	4	5	6	7							**14**		**1**	
4	5	6	7	8								**1**	**2**	
					10	11	12	13	14		**1**	**2**		
	7		**9**		11	**12**	13	14	0		**2**		**4**	
		9	**10**		12	13		0	1			**4**	**5**	
	9	**10**			13	14	0	1	2		**4**	**5**		
	10		**12**		14	0	1	2	3		**5**		**7**	

Figure 5

EXAMPLE 3.9 *In Figure* 6 *we show that the element* $(4,5;9)$ *is necessary in the partial Latin square* C_{12} *in* BC_{12}. *This is an example of Case* 3 *in Theorem* 3.7. *To construct our Latin trade we first need to construct the Latin trade* $I_{2,5}$ *using Lemma* 2.4. *We show only the last eight rows of the partial Latin square. The elements of the Latin trade are shown in bold.*

4				8	**9**	10	11				
				9	10	11	0				
				10	11	0	1				
				11	0	1	2				
								4	5	6	7
9		**11**		**1**	**2**			5	6	7	8
				2	**3**			6	7	8	9
11		**1**		**3**	**4**			7	8	9	10

Figure 6

4 Unique Completion

Here we give results on the unique completion of the partial Latin squares under consideration.

THEOREM 4.10 *The partial Latin square C_{3n} has a unique completion to the Latin square BC_{3n}. Moreover if n is odd, the partial Latin square*

$$C_{3n} \setminus \{((3n-1)/2, (3n-1)/2; 3n-1)\}$$

completes uniquely to BC_{3n}.

Proof Let L_{3n} be a Latin square that contains C_{3n} (if n is even) and $C_{3n} \setminus \{((3n-1)/2, (3n-1)/2; 3n-1)\}$ (if n is odd). We will show that L_{3n} must be the back circulant square of order $3n$. First, we partition L_{3n} into nine subarrays each of which are the intersection of n adjacent rows with n adjacent columns:

$$L_{3n} = \begin{array}{|c|c|c|} \hline Q_1 & Q_2 & Q_3 \\ \hline Q_4 & Q_5 & Q_6 \\ \hline Q_7 & Q_8 & Q_9 \\ \hline \end{array}$$

Our first claim is that there are no occurrences of entry i in cells of Q_2 for all i, $0 \leq i \leq n-1$.

Consider columns $2n$ through to $3n-1$. By the definition of C_{3n}, the cells in Q_9 contain only entries in the range n through to $3n-2$. Therefore the entries 0 through to $n-1$ must each occur n times in $Q_3 \cup Q_6$. However, entry i already occurs $i+1$ times in Q_1, and $n-(i+1)$ times in Q_5, for each i, $0 \leq i \leq n-1$. This accounts for $2n$ occurrences of each entry i in the first $2n$ rows. Thus each of these entries cannot occur in cells of Q_2.

Next we fill columns n through to $\lceil (3n-3)/2 \rceil$ in Q_2. We begin with the cell $(n-1, n)$. Entries $n-1$ through to $2n-2$ occur in row $n-1$ in Q_1. Entries $2n$ through to $3n-1$ occur in column n in Q_5. So since the entries 0 through to $n-1$ do not occur in Q_2, the entry $2n-1$ must be in the cell $(n-1, n)$. Thereafter we are forced to have the entry $2n-2$ in the cell $(n-2, n)$. We continue this process, showing that entry $i+n$ occurs in cell (i, n), for each i from $n-1$ down to 0.

Continue for columns $n+1$ through to $\lceil (3n-3)/2 \rceil$ in Q_2, each time starting with the cell in row $n-1$ and working upwards. At every stage we must have the entry $i+j$ in a cell of the form (i, j).

Now, as C_{3n} and $C_{3n} \setminus \{((3n-1)/2, (3n-1)/2; 3n-1)\}$ are both symmetric, by the same argument we can determine the entries in the cells of rows n through to $\lceil (3n-3)/2 \rceil$ in Q_4, each time having entry $i+j$ in cell (i, j).

In the previous section we used the fact that C_{3n} has a symmetric pattern about the back diagonal (the diagonal containing the entries $3n-1$).

We "flipped" C_{3n} on the axis of this diagonal to obtain a partial Latin square isotopic to C_{3n}. In other words,

$$C_{3n} = \{(3n-1-i, 3n-1-j; 3n-2-k \pmod{3n}) \mid (i,j;k) \in C_{3n}\}.$$

Exploiting this symmetry, the same argument as before determines uniquely the entries in rows $\lfloor(3n+1)/2\rfloor$ through to $2n-1$ in Q_6 and columns $\lfloor(3n+1)/2\rfloor$ through to $2n-1$ in Q_8.

We have shown that entry $i+j$ occurs in cell (i,j) whenever $i+j < \lceil(3n-1)/2\rceil$, and that entry $i+j-n$ occurs in cell (i,j) whenever $i+j \geq 3n+\lfloor(3n-1)/2\rfloor$. It follows that L_{3n} is a superset of the partial Latin square $\epsilon_{3n,\lceil(3n-1)/2\rceil}$, defined in the introduction. From Theorem 1.1, $\epsilon_{3n,\lceil(3n-1)/2\rceil}$ is a critical set in BC_{3n}; hence it has unique completion to BC_{3n}. Thus, L_{3n} must be equal to BC_{3n}. ∎

Now we are ready to state the main theorem of this paper.

THEOREM 4.11 *The partial Latin square C_{3n} is an n-balanced critical set in BC_{3n} for n even and the partial Latin square $C_{3n} \setminus \{((3n-1)/2, (3n-1)/2; 3n-1)\}$ is a near n-balanced critical set in BC_{3n} for n odd.*

Proof By Theorems 3.7 and 4.10 and the structures of C_{3n} for n even and $C_{3n} \setminus \{((3n-1)/2, (3n-1)/2; 3n-1)\}$ for n odd the result follows. ∎

The back circulant Latin square of order n, $n > 1$, can be decomposed into four disjoint critical sets (see [2]). As an application of Theorem 4.11 we have the following.

COROLLARY 4.12 *The back circulant Latin square of order $3n$, where n is even, can be decomposed into three isotopic and disjoint n-balanced critical sets.*

Proof It is easy to see that C_{3n}, $C_{3n} \oplus (n, 0)$ and $C_{3n} \oplus (2n, 0)$ are isotopic and decompose BC_{3n}. ∎

5 Computational results

Recall that a balanced critical set of order n should have size kn for some k. Now using the results in [1] one can see the following facts. Here we use the numbering system from [6] for main classes of Latin squares of small order.

1. There is only one main class for critical sets of order 3 and size 3. This critical set is not balanced. There is no critical set of order 3 and size 6 or 9.

2. There is only one main class for critical sets of order 4 and size 4. This critical set is not balanced. There is no critical set of order 4 and size 8, 12 or 16.

3. There are precisely 6 main classes for critical sets of order 5 and size 10 in main class Latin square 5.1 and there are precisely 39 main classes for critical sets of order 5 and size 10 in main class Latin square 5.2. None of these critical sets is balanced. There is no critical set of order 5 and size 15, 20 or 25.

4. Through the use of computer searches in critical sets of sizes 12 and 18 in Latin squares of order 6 we found that there does not exist any balanced critical set in main classes 6.3, 6.7, 6.8 and 6.12. We also found all the balanced critical sets in the other main classes of Latin squares of order 6. These are given below (all are 2-balanced critical sets). Note that the entries are 0, 1, 2, 3, 4 and 5.

In main class 6.1 there are precisely three balanced critical sets.

0	1				
1	2				
		4	5		
		5	0		
				2	3
				3	4

				4	5
		3			0
2	3				
	4		0		
			1	2	
5		1			

				4	5
	2	3			
			5		1
3				1	
4		0			
	0		2		

In main class 6.2 there is precisely one balanced critical set.

			3		5
1	0				
		4		0	
3		5			
			1		2
	4			2	

In main class 6.4 there are precisely four balanced critical sets.

			3		5
1	0				
		0		3	
	5				2
4				2	
		1	4		

		2	3		
	0				4
				3	1
			0	1	
4		5			
5	2				

		2	3		
	0				4
	4		5		
3				1	
				2	0
5		1			

	1		3		
				5	4
2		0			
	5		0		
4				2	
		1			3

In main class 6.5 there is precisely one balanced critical set.

		2	3		
	0				4
		4		0	
3			1		
	5				2
5				1	

In main class 6.6 there is precisely one balanced critical set.

			3		5
	0			5	
2		4			
			0	1	
4					3
	2	1			

In main class 6.9 there are precisely four balanced critical sets.

				4	5
	0		2		
	4	0			
3		1			
			1		2
5				3	

				4	5
	0	3			
2					3
		1	4		
			1	0	
5	2				

			3	4	
	0				4
2		0			
	5	1			
			1		2
5				3	

			3	4	
1	0				
2		0			
	5		4		
		5			2
				3	1

In main class 6.10 there is precisely one balanced critical set.

0					5
			2		4
	4	0			
3				1	
		5	1		
	3			2	

In main class 6.12 there are precisely two balanced critical sets.

			3		5
	0				2
		1	5		
3				2	
4				1	
	4	0			

			3	4	
	0			5	
2					4
3			1		
	2	5			
		0			1

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