Probability Models and Stochastic Processes (STAT3004)

Tutorial and Assignment questions, Weeks 1–3.

Assignment 1 consists of the *starred* ($*$) exercises, and is due on Friday 8 August. Minimal help will be given on hash (#) exercises. When attempted they may yield *bonus points* for your assignment. No points will be subtracted when not attempted.

1. (*) Let $\mathcal{F}$ be a collection of subsets of a set Ω that satisfy the conditions 1, 2, and 3 in Definition 1.3; in other words, $\mathcal{F}$ is a σ-algebra on Ω. Show that $\mathcal{F}$ is closed under countable intersections, that is, if $A_1, A_2, \ldots \in \mathcal{F}$, then so is $\cap_n A_n$.

2. (*) Let A, B, C be a *partition* of Ω — that is, the union of A, B and C is Ω and the sets do not intersect each other. Write out the *smallest* σ-algebra containing the sets A, B and C.

3. (#) Consider a σ-algebra $\mathcal{F}$ on $\mathbb{R}$ containing all intervals $(-\infty, x]$, $x \in \mathbb{R}$. Show that it must also contain the sets of the form $(a, b]$, (a, b) and $\{a\}$, and countable unions thereof.

4. Define a *probability space* for each of the following random experiments:
   (a) Roll n times with a fair die; observe all face values.
   (b) Roll a fair die until 6 comes up; observe only the number of trials required.
   (c) Toss a biased coin infinitely many times; observe whether each toss yields heads (=1) or tails (=0).
   (d) Choose at random a point in unit disk. Each point is equally likely.

5. Consider the experiment where we throw two times with a fair die.
   (a) Give a probability space for this experiment.
   (b) Construct random variables X and Y that can be interpreted as the result of the first and second throw.
   (c) Construct a random variable Z that can be interpreted as "the sum of the dice".
   (d) Specify the event $\{Z = 7\}$.
   (e) Determine the probabilities $\mathbb{P}(Z = i)$, $i = 2, \ldots, 12$.

6. We draw two cards from a full deck of 52 cards. What is the probability of drawing at least one Ace?

7. (*) Throw at random 10 balls into 4 boxes. What is the probability that exactly 2 boxes remain empty?

8. In a collection of 100 CDs there are 30 *favorites*. We select at random 20 CDs. What is the probability that 10 of them are favorites?

9. (#) Twenty husbands and wives (ten couples) are randomly divided into two groups. What is the probability that at exactly 5 wives are in the same group as their husbands?

10. (#) Let X be a random variable with $\mathbb{P}(X = k) = (1/2)^k, k = 1, 2, \ldots$. Show that such an object really exists. That is, construct and example of a probability space and a function X with the above property.

11. (∗) Let $\mathbb{P}$ be a probability measure. If $A_1, A_2, \ldots$ is a *decreasing* sequence of events, i.e., $A_1 \supset A_2 \supset \cdots$, then

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right).$$

We say that $\mathbb{P}$ is *continuous from above*.

Prove this property, using the three axioms for a probability measure and/or the *continuity from below* property.

12. (*) Let μ be the distribution of a random variable X. Show that μ is a *probability measure* on $\bar{\mathcal{B}}$. That is, verify the three axioms of probability for μ, using the collection of "events" $\bar{\mathcal{B}}$. Hint: use both the fact that $\mathbb{P}$ is a probability measure and that X is a random variable.

13. Let F be the cumulative distribution function of a random variable X. Using the three axioms of a probability measure and the definition of a random variable, show that F is increasing.

14. (#) Prove the right-continuity in Theorem 2.1 using assignment/tutorial exercise 4 and the Axioms in Definition 1.4.

15. Prove that $\Gamma(n) = (n-1)!$ and $\Gamma(1/2) = \sqrt{\pi}$.

16. Recall from STAT2003 that the *inverse-transform* method is an important device to generate random variables from a cdf F. The gist of the method is that if $U \sim \mathsf{U}(0,1)$, then $X = F^{-1}(U) \sim F$, where F^{-1} is the functional inverse of F. Prove this.

17. (*) Let

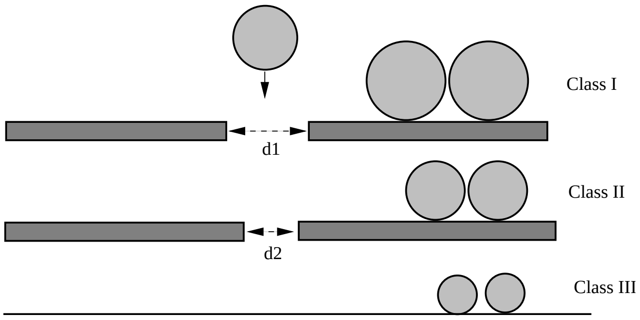
$$F(x) = \frac{1}{4}F_d(x) + \frac{3}{4}F_c(x),\ x \in \mathbb{R},$$

where F_d is the cdf of a Bernoulli random variable with success probability 1/3 and F_c is the cdf of a normal random variable with mean 1 and variance 4.

   (a) Draw the graph of the cdf F.

   (b) Write a short program to generate random variables from this distribution, using the inverse transform method.

18. (*) The diameter of a certain kind of apple is normally distributed with mean 7cm and standard deviation 1cm. We wish to sort them into three classes, using a series of sieves. Class I should contain big apples, Class II medium apples and Class III small apples, in the proportion 20%, 50% and 30%.



   (a) How should we choose the gaps $d1$ and $d2$?

   (b) If the gaps are appropriately chosen, what is the probability that out of 10 apples, 6 or more apples fall into Class II?

19. ($*$) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ be *increasing* sequences of events, that is, $A_1 \subset A_2 \subset \cdots$ and $B_1 \subset B_2 \subset \cdots$. Suppose that for each i, A_i and B_i are *independent*. Prove that the events $\cup_{i=1}^{\infty} A_i$ and $\cup_{i=1}^{\infty} B_i$ are independent as well. Clearly state what results/theorems you use in your proof. You may use the "continuity from below" property of $\mathbb{P}$.

20. We repeatedly toss a fair coin. Formulate this experiment in terms of a Bernoulli process. Let N be the first time that Heads occurs and let S be the total number of Heads in the first 10 tosses. Determine the marginal distributions of N and of S. Determine $\mathbb{P}(N = 1, S = 5)$. Are N and S independent?

21. Suppose the price of a stock during a certain year goes up with probability 0.7 and down with probability 0.3, independent of previous years.

   (a) What is the probability that the price will go up for three consecutive years?

   (b) What is the probability that the price will move in the same direction in the next two years?