THE UNIVERSITY OF QUEENSLAND

Second Semester Examination, November 2001

STAT3004

Probability Models and Stochastic Processes

(Unit Courses)

Time: TWO hours for working.
Ten minutes for perusal before examination begins.

This paper contains 7 questions which carry a total of 50 marks.

Do not remove this paper from the Examination Room

Candidates are permitted to take one A4 sheet of paper (double sided) of their own notes to the exam. No other material is permitted. Silent, hand-held pocket calculators are permitted.

1. [5 Marks] A space shuttle has two booster rockets. Each booster rocket consists of four segments. The joint between adjoining segments is sealed by a pair of O-rings. The joint is properly (safely) sealed if at least one of the two O-rings works. Let A_i be the event that the ith joint is properly sealed, $i = 1, \ldots, 6$. The (O-ring) system fails if at least one of the joints is not properly sealed.

   If the 12 O-rings fail independently of each other, each with failure probability 0.2, what is the probability of system failure?

2. [7 Marks] Let $\mathbb{P}$ be a probability measure. If $A_1, A_2, \ldots$ is a *decreasing* sequence of events, i.e., $A_1 \supset A_2 \supset \cdots$, then

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right).$$

   We say that $\mathbb{P}$ is *continuous from above.*

   Prove this property, using the three axioms for a probability measure and/or the *continuity from below* property.

3. [7 Marks] Let $X_1, X_2, \ldots$ be independent, each with an $\mathsf{Exp}(\lambda)$ (exponential) distribution. Let $N \sim G(p)$ (geometric distribution) be independent of the X_i's. Define

$$Y = X_1 + \cdots + X_N \ .$$

   (a) Determine $\mathbb{E}[Y \mid N]$ and $\mathbb{E}Y$; motivate your answer.

   (b) Show that Y has an exponential distribution, and give the corresponding parameter.

4. [7 Marks] Consider a game in which a fair coin is tossed indefinitely. Every time Heads appears you move one metre to the right, but if Tails appears you move one metre to the left. You start at position 0. Let Y_n be your position (in metres to the right of position 0) after n tosses.

   (a) Determine the expectation and variance of Y_n.

   (b) What is the probability that you are back at 0 after n tosses?

5. [8 Marks] Let $\{X_n, n = 0, 1, 2, \ldots\}$ be a Markov chain with state space $\{0, 1, 2\}$, (one-step) transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{pmatrix},$$

   and initial distribution $\pi^{(0)} = (1/2, 0, 1/2)$.

   (a) Determine

      i. $\mathbb{P}(X_2 = 2 \mid X_0 = 0)$;

      ii. $\mathbb{P}(X_1 = 2)$;

      iii. $\mathbb{E}X_1$;

      iv. $\mathbb{P}(X_0 = 2 \mid X_1 = 2)$.

   (b) Calculate the stationary distribution of $\{X_n\}$.

6. [8 Marks] Let (N_t) be a Poisson process with rate a. Calculate (in terms of a)

   (a) $\mathbb{P}(N_1 = 1, N_2 = 2)$;

   (b) $\mathbb{P}(N_4 = 3 \mid N_2 = 2)$;

   (c) $\mathbb{E}N_4$;

   (d) $\mathbb{P}(N[0, 2] = 1, N[1, 3] = 3)$.

7. [8 Marks] An airoplane has two identical computers that control simultaneously the flight of the plane. Both computers can fail (independently of each other). The life times have an exponential distribution with expectation $1/a = 10$ (hours). There are two technicians on the plane. They cannot jointly work on a failed computer. The repair time of a computer has an exponential distribution with expectation $1/b = 1$ (hour). Repair times and life times are independent. Let X_t denote the number of failed computers at time t (hours). Assume $X_0 = 0$. (X_t) is a Markov process. We say that the system is down when both computers are down.

   (a) Sketch a realization of (X_t).

   (b) Give the transition rate graph and the generator (=Q-matrix) of (X_t).

   (c) Determine the limiting probability that the system is down.

   (d) Suppose at time t the system has been down for 1 hour. What is the probability that the system will be down for an extra 1 hour?

END OF PAPER