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Second-order asymptotics in level crossing for differences of renewal processes

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We consider level crossing for the difference of independent renewal processes. Second-order expansions for the distribution function of the crossing time of level n are found, as $n \to \infty$. As a by-product several other results on the difference process are found. The expected minimum of the difference process appears to play an important role in the analysis. This makes this problem essentially harder than the level crossing for the sum process which was studied earlier.

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difference of renewal processes * boundary crossing * second-order approximations

1. Introduction

Although renewal processes are among the most important and basic processes in applied probability, various areas in renewal theory are still rather unexploited. One of these more or less "blank spots on the map" seems to be the *difference* process of two independent renewal processes, which occurs frequently in queueing and reliability theory. Let $N_1 = (N_1(t))$ and $N_2 = (N_2(t))$ be two independent renewal processes and let the difference process M be defined by $M(t) = N_1(t) - N_2(t)$, $t \geqslant 0$. Assume that M has an upward drift. We are in particular interested in the crossing time τ_n of level $n \in \mathbb{N}$. The situation resembles in a way the model of Kroese and Kallenberg (1989) [KK] where level crossing was considered for the *sum* process of independent renewal processes. For that model *second-order* approximations to the distribution function (d.f.) of the crossing time of level n were found, as $n \to \infty$. These approximations depend only on the first three moments of the d.f.'s defining the renewal sequences.

Both for the sum process and for the difference process a first-order approximation, based on asymptotic normality, can be easily established using Cox (1962, p. 73). However, as seen in [KK], first order approximations are not very accurate and a big improvement is obtained by applying second-order approximations, leading to quite satisfactory approximations even for very small values of n.

The main result of the paper is a second-order expansion for the d.f. of τ_n, when suitably standardized, as $n \to \infty$. In the derivation of this expansion we come across several results, which are of independent interest. First of all, in Theorem 4.2 the d.f. of $M(t)$ for large values of t is approximated in a similar way as the d.f. of the sum process, using expansions for the individual processes. In the case of the sum process a classical inversion step makes it possible to give a second-order expansion for the d.f. of the crossing time of level n. However, for the difference process, such a relatively simple inversion step is not available, due to the fact that in this situation the process is not increasing. To tackle these complications we rely on an idea of Anscombe (1953) to "look backwards in time". This idea is most easily used when both N_1 and N_2 are *stationary* renewal processes. In the recent past the idea of Anscombe has been applied frequently in boundary crossing problems for *random walks* (cf. Woodroofe (1982), Lalley (1984), Siegmund (1985) and Woodroofe and Keener (1986)).

Apart from a term which may be seen as the equivalent of the inversion step in the expansion for the sum process (cf. (3.2) and p. 486 of [KK]) now an extra term comes in, where the expectation of the minimum of M emerges. It is shown that this expectation exists finitely, due to the positive drift of M and the finiteness of the second moments of the corresponding renewal sequences. Finally a delicate investigation of the local behavior of M at time t for large t is given for the finishing touch.

From a theoretical point of view the second-order expansion of the d.f. of τ_n gives insight in the structure of the problem. The approximation does not only depend on the first three moments of the d.f.'s defining the renewal sequences. The somewhat unexpected appearance of the expected minimum of M in the expansion shows that the d.f. of the crossing time for the difference process is essentially more complicated than for the sum process.

From a computational point of view, (3.1) shows that for a second-order approximation to the d.f. of τ_n it is not sufficient to know the first three moments of the d.f.'s involved. Fortunately, in many practical cases the expected minimum of M can be derived, whereas the true d.f. of τ_n is practically always intractable. Numerical results and special cases are given in Kroese (1992). If the positive drift is not too small, the approximations are close to the (estimated) true d.f. even for rather small n.

We conclude the Introduction with an outline of the rest of the paper. In the next section the basic definitions and model assumptions are given. Section 3 lists the main result and sketches its proof. The formal proof of the main result is given in Sections 4–5.

2. Model, definitions

In this section we formalize the model. Throughout this paper we conform to the definitions and assumptions that are given here, unless otherwise specified. There exist several definitions of a renewal process. Here we use the following:

Let $X=(X_i)_{i\geq 1}$ be a sequence of *independent nonnegative* random variables, such that $X_2, X_3, \ldots$ are *identically distributed* with some d.f. G. Let F denote the d.f. of X_1. We call X the *renewal sequence* corresponding to the *delayed renewal process* $N=(N(t))_{t\geq 0}$, which is defined for $t\geq 0$ by

$$N(t)=\begin{cases}0, & \text{if } X_1>t,\\ \sup\{n\geq 1\colon X_1+\cdots+X_n\leq t\}, & \text{else.}\end{cases}$$

N is called an *ordinary* renewal process if $F=G$, and a *stationary* renewal process if

$$F(x)=\frac{1}{\mu}\int_0^x \{1-G(y)\}\,\mathrm{d}y, \quad \text{where } \mu=EX_2<\infty.$$

Let $N_1=(N_1(t))$ and $N_2=(N_2(t))$ denote two *independent* renewal processes with renewal sequences $(X_k^{(1)})$ and $(X_k^{(2)})$, respectively. If we omit the index set in the definition of a stochastic process, it is either $\mathbb{N}_+$ (renewal sequences) or $\mathbb{R}_+$ (otherwise). The difference process $M=(M(t))$ of N_1 and N_2 is defined by

$$M(t)=N_1(t)-N_2(t), \quad t\geq 0.$$

Let τ_n be the time at which M crosses level $n\in\mathbb{N}$, i.e.

$$\tau_n=\inf\{t\geq 0\colon M(t)\geq n\}.$$

Similarly, let T_k be the first time at which M crosses level $k\in\mathbb{Z}_-$,

$$T_k=\inf\{t\geq 0\colon M(t)\leq k\}.$$

We do not a priori assume N_1 and N_2 to be stationary renewal processes. Although the main result is stated for such processes, most of the results hold true for delayed renewal processes. Let F_i be the d.f. of $X_1^{(i)}$ and let G_i be the d.f. of $X_2^{(i)}$, $i=1,2$. We assume that the *expectation* μ_i, the *variance* σ_i^2 and the *third central moment* μ_{3i} of $X_2^{(i)}$ exist *finitely* and that G_i is *nonlattice*, $i=1,2$. This also ensures the finiteness of the expectation and variance of $X_1^{(i)}$ when N_i is stationary, $i=1,2$. Let

$$\zeta_i=\sup\{x\geq 0\colon F_i(x)<1\}, \quad i=1,2.$$

The probability space in the background is denoted by $(\Omega, \mathscr{H}, P)$. We introduce also a family of probability measures $(P^{r,s};\, 0\leq r<\zeta_1, 0\leq s<\zeta_2)$ on $(\Omega, \mathscr{H})$. Under measure $P^{r,s}$, N_1 and N_2 are independent renewal processes, such that $X_2^{(i)}$ has d.f. G_i, $i=1,2$, and, for $x, y\geq 0$,

$$P^{r,s}(X_1^{(1)}\leq x, X_1^{(2)}\leq y)=\frac{F_1(x+r)-F_1(r)}{1-F_1(r)}\,\frac{F_2(y+s)-F_2(s)}{1-F_2(s)}.$$

The expectation operator corresponding to $P^{r,s}$ is denoted by $E^{r,s}$.

Remark 2.1. When $F_i(0)=0$, for $i=1,2$ (as is the case for stationary N_1 and N_2), then probability measures P and $P^{0,0}$ coincide.

The following notation for conditional probabilities is used. Let $(\Omega, \mathscr{H}, Q)$ be the probability space under use. Let H be an event and let $\mathscr{G}$ be a sub σ-algebra of $\mathscr{H}$.

The conditional probability of the event H, given $\mathcal{G}$ is denoted by $Q_{\mathcal{G}}(H)$. If $\mathcal{G}$ is generated by a random variable V, we write $Q_V(H)$ for $Q_{\mathcal{G}}(H)$.

We assume from now on that $\mu_1 < \mu_2$. In that case M has a positive drift α^{-1},

$$\alpha^{-1} = \mu_1^{-1} - \mu_2^{-1}.$$

The following parameters are used frequently in the expansions below. We therefore list them in this section. We define

$$\gamma = \sqrt{\sigma_1^2 \mu_1^{-3} + \sigma_2^2 \mu_2^{-3}}$$

and, for $x \in \mathbb{R}$,

$$t_n = t_n(x) = n\alpha + x\gamma\sqrt{n}\,\alpha^{3/2},$$

$$c_i = \tfrac{1}{6}\mu_{3i}\sigma_i^{-3}\mu_i^{1/2}, \quad d_i = \tfrac{1}{2}\sigma_i\mu_i^{-1/2}, \quad a_i = \mu_i^{3/2}\sigma_i^{-1}, \qquad i = 1, 2. \tag{2.1}$$

The standard normal d.f. is denoted by Φ, its density by φ. To conclude this section, Figure 1 gives an illustration of the various definitions (note that $k \leq 0$).

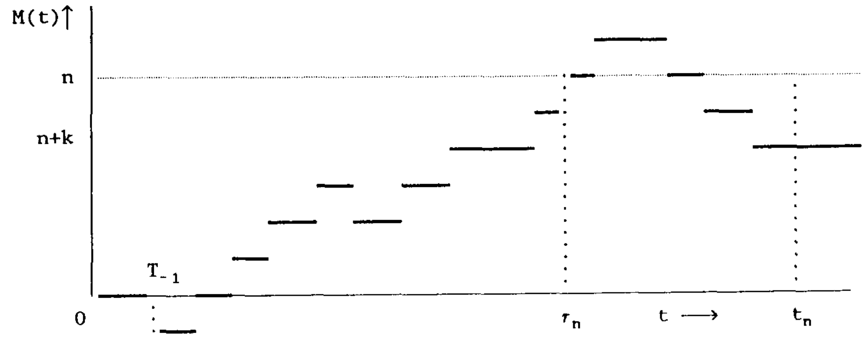

Fig. 1. Realization of $(M(t), t \geq 0)$.

3. Main result

Theorem 3.1. *Let N_1 and N_2 be independent stationary renewal processes that satisfy the conditions of Section 2. Then the following second-order expansion for τ_n holds true.*

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{\tau_n - n\alpha}{\gamma\alpha^{3/2}\sqrt{n}} \leq x\right) - \Phi(x) \right.$$
$$\left. - \frac{\varphi(x)}{\sqrt{n}} \left\{ p(1-x^2) + \frac{1}{2\gamma\sqrt{\alpha}} - \tfrac{1}{2}\gamma\sqrt{\alpha} - \frac{1}{\gamma\sqrt{\alpha}} E \inf_{t\geq 0} M(t) \right\} \right|$$
$$= o(1/\sqrt{n}) \quad \textit{as } n \to \infty, \tag{3.1}$$

where

$$p = \frac{(c_1 - d_1)(\gamma a_1)^{-3} - (c_2 - d_2)(\gamma a_2)^{-3}}{\sqrt{\alpha}} + \tfrac{1}{2}\gamma\sqrt{\alpha}.$$

(The other parameters are defined in Section 2.)

Remark 3.1. First we show that the terms in (3.1) are well-defined (i.e. that $\inf_{t\geq 0} M(t)$ has finite expectation). This can be seen as follows. Let $S_n^{(i)} = X_1^{(i)} + \cdots + X_n^{(i)}$, $i = 1, 2$. For $k \in \mathbb{N}_+$,

$$\begin{aligned} P\Big(-\inf_{t\geq 0} M(t) \geq k\Big) &= P(S_n^{(1)} > S_{n+k-1}^{(2)} \text{ for some } n) \\ &\leq P\Big(\sup_{j\geq k} j^{-1}|S_j^{(2)} - j\mu_2| > \varepsilon\Big) + P(S_n^{(1)} > (n+k-1)(\mu_2 - \varepsilon) \text{ for some } n) \\ &\leq P\Big(\sup_{j\geq k} j^{-1}|S_j^{(2)} - j\mu_2| > \varepsilon\Big) \\ &\quad + P\Big(\sup_{n\geq 0} \{S_n^{(1)} - n(\mu_2 - \varepsilon)\} > (\mu_2 - \varepsilon)(k-1)\Big), \end{aligned}$$

where $\varepsilon > 0$ is so small that $\mu_1 < \mu_2 - \varepsilon$. Now

$$\begin{aligned} -E \inf_{t\geq 0} M(t) &= \sum_{k=1}^{\infty} P\Big(-\inf_{t\geq 0} M(t) \geq k\Big) \\ &\leq \sum_{k=1}^{\infty} P\Big(\sup_{j\geq k} j^{-1}|S_j^{(2)} - j\mu_2| > \varepsilon\Big) \\ &\quad + (\mu_2 - \varepsilon)^{-1} E \sup_{n\geq 0} \{S_n^{(1)} - n(\mu_2 - \varepsilon)\} + 1. \end{aligned}$$

The sum in the previous line is finite by Theorem 3 of Baum and Katz (1965), while the second expression is finite e.g. by Theorem 5 of Kiefer and Wolfowitz (1956).

Next we give a broad sketch of the proof of Theorem 3.1, of which the details can be found in the remaining sections.

Sketch of the proof of Theorem 3.1. By definition of t_n and τ_n,

$$\begin{aligned} P\left(\frac{\tau_n - n\alpha}{\gamma\alpha^{3/2}\sqrt{n}} \leq x\right) &= P(\tau_n \leq t_n) \\ &= P(M(t_n) \geq n) + P(M(t_n) < n, \tau_n \leq t_n). \end{aligned} \tag{3.2}$$

We tackle the first term in (3.2) by using techniques that were developed in [KK]. That is, first we derive an expression for $P(N_i(t) \leq x)$, as $t \to \infty$, $i = 1, 2$. Then, in order to obtain an expansion for $P(M(t) \leq x)$ it suffices to convolve the expansions of $P(N_1(t) \leq x)$ and $P(N_2(t) \leq x)$. This first term may be seen as the equivalent of the inversion step in the expansion for the sum process. For the second term in (3.2) we have,

$$P(M(t_n)<n,\ \tau_n\leqslant t_n)$$
$$=\sum_{k=-\infty}^{-1} P(M(t_n)=n+k,\ \tau_n\leqslant t_n)$$
$$=\sum_{k=-\infty}^{-1} P\Big(M(t_n)-M(t_n-t_n)=n+k,\ \min_{0\leqslant u\leqslant t_n}\{M(t_n)-M(t_n-u)\}\leqslant k\Big).$$

Since N_1 and N_2 are *independent stationary* renewal processes, $(M(t_n)-M(t_n-u))_{0\leqslant u\leqslant t_n}$ has the same probability law as $(M(u))_{0\leqslant u\leqslant t_n}$. This is a direct consequence of the reversibility property of stationary renewal sequences and Anscombe's (1953) idea to "look backwards in time". This idea is frequently used in level crossing problems for random walks (cf. Section 1), and is most easily illustrated by turning the realization of M in Figure 1 upside down and viewing the process "backwards in time", starting at time t_n. Hence by definition of T_k in Section 2, we have

$$P(M(t_n)<n,\ \tau_n\leqslant t_n)=\sum_{k=-\infty}^{-1} P(M(t_n)=n+k,\ T_k\leqslant t_n). \tag{3.3}$$

We will show that, up to $o(n^{-1/2})$, we may restrict attention to a finite sum in (3.3) and that the summand may be replaced by

$$P(M(t_n)=n+k)\cdot P(T_k\leqslant t_n).$$

Moreover,

$$P(M(t_n)=n+k)\approx\varphi(x)\gamma^{-1}(n\alpha)^{-1/2}$$

for fixed k and large enough n, and finally

$$\sum_{k=-\infty}^{-1} P(T_k\leqslant t_n)\approx\sum_{k=-\infty}^{-1} P(T_k<\infty)=-E\inf_{t\geqslant 0} M(t).$$

The approximation sign $\approx$ will be made more precise in the next sections, where we will give the formal proof of Theorem 3.1. The first term in (3.2) is considered in Section 4, whereas the second part is treated in Section 5. The actual proof of the theorem is given at the end of Section 5.

4. Expansions, first part

In order to expand the first term in (3.2), $P(M(t_n)\geqslant n)$, for large n, we start with an expansion for $P(M(t)\leqslant x)$ for large t.

Throughout this section $N=(N(t))$ denotes a delayed renewal process with renewal sequence (X_i). Let F be the d.f. of X_1 and G the d.f. of X_i, $i=2,3,\ldots$.

We assume that F and G have finite expectation, variance and third central moment and that G is nonlattice. Let $\zeta=\sup\{x\geqslant 0\colon\ F(x)<1\}$. The probability measure $P^r, 0\leqslant r<\zeta$, is defined in the same manner as $P^{r,s}$ in Section 2.

Let $\mu=EX_2$, $\sigma^2=\operatorname{Var} X_2$, $\mu_3=E(X_2-\mu)^3$ and $\nu=\nu(r)=E^rX_1, 0\leqslant r<\zeta$. Define parameters a, c and d similarly to (2.1). Note that $\nu(0)$ is not necessarily equal to EX_1, cf. Remark 2.1. We have the following lemma.

Lemma 4.1. *Let $S_n=X_1+\cdots+X_n$. For every fixed $0\leqslant y<\zeta$,*

$$\sup_{0\leqslant r\leqslant y}\sup_{x\in\mathbb{R}}\left|P^r\left(\frac{S_n-(n-1)\mu-\nu}{\sigma\sqrt{n}}\leqslant x\right)-\Phi(x)-\frac{\varphi(x)\mu_3}{\sqrt{n}\,6\sigma^3}(1-x^2)\right|$$
$$=\mathrm{o}(n^{-1/2})\quad \textit{as } n\to\infty.$$

Proof. Let $\tilde{S}_{n-1}=X_2+\cdots+X_n$ and $Y_n=(\sigma x\sqrt{n}-X_1+\nu)\sigma^{-1}(n-1)^{-1/2}$, then

$$P^r\left(\frac{S_n-(n-1)\mu-\nu}{\sigma\sqrt{n}}\leqslant x\right)=P^r\left(\frac{\tilde{S}_{n-1}-(n-1)\mu}{\sigma\sqrt{n-1}}\leqslant Y_n\right)$$
$$=E^rP^r_{X_1}\left(\frac{\tilde{S}_{n-1}-(n-1)\mu}{\sigma\sqrt{n-1}}\leqslant Y_n\right).$$

Consequently,

$$\left|P^r\left(\frac{S_n-(n-1)\mu-\nu}{\sigma\sqrt{n}}\leqslant x\right)-\Phi(x)-\frac{\varphi(x)\mu_3}{\sqrt{n}6\sigma^3}(1-x^2)\right|$$
$$\leqslant E^r\left|P^r_{X_1}\left(\frac{\tilde{S}_{n-1}-(n-1)\mu}{\sigma\sqrt{n-1}}\leqslant Y_n\right)-\Phi(Y_n)-\frac{\varphi(Y_n)\mu_3}{\sqrt{n-1}\,6\sigma^3}(1-Y_n^2)\right| \tag{4.1}$$
$$+\left|E^r\Phi(Y_n)-\Phi(x)\right| \tag{4.2}$$
$$+E^r\left|\frac{\varphi(Y_n)\mu_3}{\sqrt{n-1}\,6\sigma^3}(1-Y_n^2)-\frac{\varphi(x)\mu_3}{\sqrt{n}\,6\sigma^3}(1-x^2)\right|. \tag{4.3}$$

It is clear that (4.1) is of the order $\mathrm{o}(n^{-1/2})$ uniformly in x and r (cf. Feller, 1971, Theorem XVI 4.1, p. 539). Moreover, for $|x|=|x_n|\leqslant\log n$,

$$Y_n-x=\frac{\nu-X_1}{\sigma\sqrt{n-1}}+\mathrm{O}((n-1)^{-1}\log n).$$

It follows from the mean value theorem that both (4.2) and (4.3) are $\mathrm{o}(n^{-1/2})$ uniformly for $0\leqslant r\leqslant y$ and $|x|\leqslant\log n$. In particular,

$$\sup_{0\leqslant r\leqslant y}P^r\left(\frac{S_n-(n-1)\mu-\nu}{\sigma\sqrt{n}}\leqslant -\log n\right)=\mathrm{o}(n^{-1/2})$$

and

$$\sup_{0\leqslant r\leqslant y}P^r\left(\frac{S_n-(n-1)\mu-\nu}{\sigma\sqrt{n}}\leqslant \log n\right)=1-\mathrm{o}(n^{-1/2}),$$

so that Lemma 4.1 follows by the monotonicity of d.f.'s. $\square$

We now have the means to derive an expansion for the d.f. of the delayed renewal process $N(t)$ starting with 'age' r.

Theorem 4.1. *For $0 \leq y < \zeta$ fixed,*

$$\sup_{0 \leq r \leq y} \sup_{x \in \mathbb{R}} \left| P^r\left(\frac{N(t)-t/\mu}{\sqrt{t}/a} \leq x\right) - \Phi(x) \right.$$
$$\left. - \frac{\varphi(x)}{\sqrt{t}} \left\{ (c-d)(x^2-1) + e + a\left\{ \theta\left(\frac{t}{\mu} + xa^{-1}\sqrt{t}\right) - \tfrac{1}{2} \right\} \right\} \right|$$
$$= \mathrm{o}(t^{-1/2}) \quad \text{as } t \to \infty,$$

where $e = -d - \frac{1}{2}a + \mu^{1/2}\sigma^{-1}\nu$ and $\theta(y) = [y] + 1 - y$ with $[y]$ the integer part of y.

Proof. Let $|x| = |x_t| \leq \log t$.

$$P^r\left(\frac{N(t)-t/\mu}{\sqrt{t}/a} \leq x\right) = P^r(N(t) \leq t/\mu + xa^{-1}\sqrt{t})$$
$$= P^r(N(t) \leq m_t - \theta(t/\mu + xa^{-1}\sqrt{t})) = P^r(S_{m_t} > t),$$

where

$$m_t = [t/\mu + xa^{-1}\sqrt{t}] + 1.$$

Let $y_t = (t - (m_t - 1)\mu - \nu)\sigma^{-1}m_t^{-1/2}$, then by Lemma 4.1,

$$P^r(S_{m_t} > t) = 1 - \Phi(y_t) - \frac{\varphi(y_t)\mu_3}{\sqrt{m_t}\, 6\sigma^3}(1 - y_t^2) + \mathrm{o}(m_t^{-1/2}),$$

uniformly for $0 \leq r \leq y$. Moreover,

$$y_t = -x + \frac{x^2\sigma\mu^{-1/2}}{2\sqrt{t}} + \frac{\mu^{1/2}(\mu-\nu)}{\sigma\sqrt{t}} - \frac{\mu^{3/2}}{\sigma\sqrt{t}}\theta(t/\mu + xa^{-1}\sqrt{t})$$
$$+ \mathrm{O}(t^{-1}\log^3 t),$$

uniformly for $0 \leq r \leq y$, $|x| \leq \log t$. Therefore, by the mean value theorem and the finiteness of ν, we have,

$$P^r\left(\frac{N(t)-t/\mu}{\sqrt{t}/a} \leq x\right) = \Phi(x) - \frac{\varphi(x)}{\sqrt{t}}c(1-x^2)$$
$$- \frac{\varphi(x)}{\sqrt{t}}\left\{ x^2 d + a - \frac{\mu^{1/2}\nu}{\sigma} - a\theta(t/\mu + xa^{-1}\sqrt{t}) \right\}$$
$$+ \mathrm{o}(t^{-1/2}),$$

uniformly for $0 \leq r \leq y$, $|x| \leq \log t$. The uniformity for $x \in \mathbb{R}$ now follows in the usual way from the monotonicity of d.f.'s. □

Corollary 4.1. *Let N_1 and N_2 be delayed renewal processes and let e_1 and e_2 be defined similarly as e in Theorem* 4.1. *For all $0 \leq y < \zeta_1$ and $0 \leq z < \zeta_2$ we have,*

$$\sup_{0\leq r\leq y}\ \sup_{0\leq s<\zeta_2}\ \sup_{x\in\mathbb{R}} \left| P^{r,s}\left(\frac{N_1(t)-t/\mu_1}{\gamma\sqrt{t}} \leq x\right) - \Phi(\gamma a_1 x) - \frac{\varphi(\gamma a_1 x)}{\sqrt{t}}\left\{(c_1-d_1)\{(\gamma a_1 x)^2-1\}+e_1 + a_1\left\{\theta\left(\frac{t}{\mu_1}+\gamma x\sqrt{t}\right)-\tfrac{1}{2}\right\}\right\}\right| = \mathrm{o}(t^{-1/2}) \tag{4.4}$$

and

$$\sup_{0\leq r<\zeta_1}\ \sup_{0\leq s\leq z}\ \sup_{x\in\mathbb{R}} \left| P^{r,s}\left(\frac{-N_2(t)+t/\mu_2}{\gamma\sqrt{t}} \leq x\right) - \Phi(\gamma a_2 x) + \frac{\varphi(\gamma a_2 x)}{\sqrt{t}}\left\{(c_2-d_2)\{(\gamma a_2 x)^2-1\}+e_2 - a_2\left\{\theta\left(\frac{-t}{\mu_2}+\gamma x\sqrt{t}\right)-\tfrac{1}{2}\right\}\right\}\right| = \mathrm{o}(t^{-1/2}). \tag{4.5}$$

Proof. (4.4) is a direct consequence of Theorem 4.1 by substituting $\gamma a_1 x$ for x. (4.5) is obtained in the same way after observing that $\lim_{h\downarrow 0}\theta(z-h)=1-\theta(-z)$, for all $z\in\mathbb{R}$. □

The following theorem is proved in exactly the same way as Theorem 4.1 of [KK], using Corollary 4.1. Since in Theorem 4.2 $\sup_{0\leq r\leq y}|e_1|<\infty$ and $\sup_{0\leq s\leq z}|e_2|<\infty$, the uniformity in r and s follows easily.

Theorem 4.2. *Let $M(t)=N_1(t)-N_2(t)$, where N_1 and N_2 are independent delayed renewal processes. Let e_1 and e_2 be defined as in Corollary* 4.1, *then for fixed $0\leq y<\zeta_1$ and $0\leq z<\zeta_2$,*

$$\sup_{0\leq r\leq y}\ \sup_{0\leq s\leq z}\ \sup_{x\in\mathbb{R}} \left| P^{r,s}\left(\frac{M(t)-t/\alpha}{\gamma\sqrt{t}} \leq x\right) - \Phi(x) - \frac{\varphi(x)}{\sqrt{t}}\left\{q(x^2-1)+\tilde{e}+\frac{1}{\gamma}\left\{\theta\left(\frac{t}{\alpha}+\gamma x\sqrt{t}\right)-\tfrac{1}{2}\right\}\right\}\right| = \mathrm{o}(t^{-1/2}),$$

as $t\to\infty$, where

$$q=(c_1-d_1)(\gamma a_1)^{-3}-(c_2-d_2)(\gamma a_2)^{-3}$$

and

$$\tilde{e}=e_1(\gamma a_1)^{-1}-e_2(\gamma a_2)^{-1}. \qquad \square$$

Corollary 4.2. *If M is the difference of two independent stationary renewal processes, then*

$$\sup_{x\in\mathbb{R}}\left|P(M(t_n)\geq n)-\Phi(x)\right.$$
$$\left.-\frac{\varphi(x)}{\sqrt{n}}\left\{\left(\frac{q}{\sqrt{\alpha}}+\tfrac{1}{2}\alpha^{1/2}\gamma\right)(1-x^2)+\frac{1}{2\gamma\sqrt{\alpha}}-\tfrac{1}{2}\alpha^{1/2}\gamma\right\}\right|=\mathrm{o}(n^{-1/2})$$

as $n\to\infty$, where q is defined in Theorem 4.2 and the other parameters in Section 2.

Proof. First let $|x|\leq \log n$. We have

$$P(M(t_n)\geq n)=1-P\left(\frac{M(t_n)-t_n/\alpha}{\gamma\sqrt{t_n}}\leq y_n\right),$$

where, by the choice of x,

$$y_n=\frac{n-1-t_n/\alpha}{\gamma\sqrt{t_n}}=-x+\frac{x^2\alpha^{1/2}\gamma}{2\sqrt{n}}-\frac{1}{\gamma\sqrt{n\alpha}}+\mathrm{O}(n^{-1}\log^3 n). \tag{4.6}$$

Observe that in this stationary case, $\nu_i=\frac{1}{2}(\mu_i+\sigma_i^2/\mu_i)$, so that $e_1=e_2=\tilde{e}=0$ in Theorem 4.2. Moreover, the argument of θ in Theorem 4.2 now equals $n-1$, and thus the θ-part simply becomes 1. Consequently,

$$P(M(t_n)\geq n)=1-\Phi(y_n)-\frac{\varphi(y_n)}{\sqrt{t_n}}\left\{q(y_n^2-1)+\frac{1}{2\gamma}\right\}+\mathrm{o}(t_n^{-1/2}),$$

so that by (4.6),

$$P(M(t_n)\geq n)=\Phi(x)-\frac{\varphi(x)}{\sqrt{n\alpha}}\left\{q(x^2-1)+\frac{1}{2\gamma}\right\}$$
$$-\frac{\varphi(x)}{\sqrt{n}}\left\{\tfrac{1}{2}x^2\gamma\sqrt{\alpha}-\frac{1}{\gamma\sqrt{\alpha}}\right\}+\mathrm{o}(n^{-1/2}),$$

as $n\to\infty$, uniformly for $|x|\leq\log n$. The corollary follows by the monotonicity of d.f.'s. □

A *local* expansion for the distribution of M is given in the next lemma. Although the result below can be extended in several ways, we only list it in the form that is sufficient for our purposes. Note that we again do not specify the distribution of the first component of N_1 and N_2.

Lemma 4.2. *Let $y\geq 0$ be fixed and let $r_0<\min\{y,\zeta_1\}$, then*

$$\sup_{0\leq r\leq r_0}\ \sup_{0\leq u\leq y}\ \sup_{|x|\leq\log n}\left|P^{r,0}(M(t_n-u)=n)-\frac{\varphi(x)}{\gamma\sqrt{n\alpha}}\right|=\mathrm{o}(n^{-1/2}).$$

Proof. Let $y_n=(n-(t_n-u)/\alpha)\gamma^{-1}(t_n-u)^{-1/2}$ and $z_n=y_n-\gamma^{-1}(t_n-u)^{-1/2}$, then

$$\begin{aligned}
&P^{r,0}(M(t_n-u)=n)\\
&\quad=P^{r,0}(M(t_n-u)\leqslant n)-P^{r,0}(M(t_n-u)\leqslant n-1)\\
&\quad=P^{r,0}\left(\frac{M(t_n-u)-(t_n-u)/\alpha}{\gamma\sqrt{t_n-u}}\leqslant y_n\right)\\
&\qquad-P^{r,0}\left(\frac{M(t_n-u)-(t_n-u)/\alpha}{\gamma\sqrt{t_n-u}}\leqslant z_n\right).
\end{aligned}$$

Since $t_n=n\alpha+x\gamma\sqrt{n}\,\alpha^{3/2}$,

$$|y_n-z_n|=\gamma^{-1}(t_n-u)^{-1/2}=\gamma^{-1}(n\alpha)^{-1/2}+\mathrm{O}(n^{-1}\log n), \tag{4.7}$$

uniformly for $0\leqslant r\leqslant r_0$, $0\leqslant u\leqslant y$ and $|x|\leqslant\log n$. Also,

$$y_n=-x+\mathrm{O}(n^{-1/2}\log^2 n), \tag{4.8}$$

uniformly for $0\leqslant r\leqslant r_0$, $0\leqslant u\leqslant y$ and $|x|\leqslant\log n$. Now

$$|P^{r,0}(M(t_n-u)=n)-\varphi(x)\gamma^{-1}(n\alpha)^{-1/2}|$$

$$\leqslant\left|P^{r,0}\left(\frac{M(t_n-u)-(t_n-u)/\alpha}{\gamma\sqrt{t_n-u}}\leqslant y_n\right)-\Phi(y_n)-\frac{\varphi(y_n)}{\sqrt{t_n-u}}p(y_n)\right| \tag{4.9}$$

$$+\left|P^{r,0}\left(\frac{M(t_n-u)-(t_n-u)/\alpha}{\gamma\sqrt{t_n-u}}\leqslant z_n\right)-\Phi(z_n)-\frac{\varphi(z_n)}{\sqrt{t_n-u}}p(z_n)\right| \tag{4.10}$$

$$+|\Phi(y_n)-\Phi(z_n)-\varphi(x)\gamma^{-1}(n\alpha)^{-1/2}| \tag{4.11}$$

$$+\left|\frac{\varphi(y_n)}{\sqrt{t_n-u}}p(y_n)-\frac{\varphi(z_n)}{\sqrt{t_n-u}}p(z_n)\right|, \tag{4.12}$$

where $p(x)=q(x^2-1)+\tilde{e}+\frac{1}{2}\gamma^{-1}$ (q and $\tilde{e}$ are defined in Theorem 4.2). By Theorem 4.2 it follows that (4.9) and (4.10) are of order $\mathrm{o}((t_n-u)^{-1/2}))=\mathrm{o}(n^{-1/2})$ uniformly for $0\leqslant r\leqslant r_0$, $0\leqslant u\leqslant y$ and $|x|\leqslant\log n$. The same result holds for (4.11) and (4.12), by the mean value theorem and (4.7) and (4.8), which completes the proof. □

The following lemma provides additional information about the local behavior of a delayed renewal process.

Lemma 4.3. *Let* $(N(t))$ *be a delayed renewal process as defined at the beginning of this section, then*

$$\sup_{t\geqslant 0}\sup_{0\leqslant r<\zeta}P^r(N(t)=n)=\mathrm{O}(n^{-1/2}).$$

Proof. Throughout the proof C denotes a generic constant. Let (X_n) be the renewal sequence of N, and define

$$\tilde{S}_{n-1} = (X_2 + \cdots + X_n - (n-1)\mu)(n-1)^{-1/2}\sigma^{-1}, \quad n \geqslant 2.$$

Then

$$P^r(N(t) = n) = P^r(\tilde{S}_{n-1} \leqslant Y) - P^r(\tilde{S}_n \leqslant Z),$$

where $Y = (t - X_1 - (n-1)\mu)\sigma^{-1}(n-1)^{-1/2}$ and $Z = (t - X_1 - n\mu)\sigma^{-1}n^{-1/2}$. Therefore,

$$\begin{aligned} P^r(N(t) = n) \leqslant\, & E^r|\Phi(Y) - \Phi(Z)| + E^r|P^r_{X_1}(\tilde{S}_{n-1} \leqslant Y) - \Phi(Y)| \\ & + E^r|P^r_{X_1}(\tilde{S}_n \leqslant Z) - \Phi(Z)|. \end{aligned} \tag{4.13}$$

By the Berry–Esseen inequality and the independence of $X_1, X_2, \ldots$ we have

$$\sup_{y \in \mathbb{R}} |P^r_{X_1}(\tilde{S}_{n-1} \leqslant y) - \Phi(y)| \leqslant \frac{C}{\sqrt{n}},$$

independently of r, so that the last two terms in (4.13) are $O(n^{-1/2})$ uniformly in t and $0 \leqslant r < \zeta$. The same holds for the first term in the right-hand side of (4.13), since it can be established by

$$\sup_{x \in \mathbb{R}} \left| \Phi\left(\frac{x}{\sqrt{n-1}}\right) - \Phi\left(\frac{x}{\sqrt{n}} - \frac{\mu}{\sigma\sqrt{n}}\right) \right| = O(n^{-1/2}).$$

This completes the proof. □

5. Expansions, second part; proof of main result

Throughout this section, M denotes the difference of two independent *stationary* renewal processes. In (3.3) we found

$$P(M(t_n) < n, \tau_n \leqslant t_n) = \sum_{k=-\infty}^{-1} P(M(t_n) = n + k, T_k \leqslant t_n).$$

We first show that it suffices to consider $P(M(t_n) = n + k,\ T_k \leqslant t_n)$ for fixed k.

Lemma 5.1. *Let M and T_k be defined as in Section* 2, *then*

$$\lim_{A \to \infty} \lim_{n \to \infty} \sup_{|x| \leqslant \log n} \sqrt{n} \sum_{k=-\infty}^{-A} P(M(t_n) = n + k,\ T_k \leqslant t_n) = 0.$$

Proof. By using conditional expectation with respect to the σ-algebra $\mathscr{F}_{T_k}$ generated by the stopping time T_k we obtain

$$P(M(t_n) = n + k,\ T_k \leqslant t_n) = EI_{\{T_k \leqslant t_n\}} P_{\mathscr{F}_{T_k}}(M(t_n) = n + k). \tag{5.1}$$

Let $Q^{r,s}$ denote the probability measure $P^{r,s}$ under which both N_1 and N_2 are *ordinary* renewal processes and $X_2^{(i)}$ has d.f. G_i, $i=1,2$. Denote ζ_1 in this case by ζ_1^*. On the other hand let the probability measure $\tilde{Q}^{r,s}$ correspond to $P^{r,s}$ such that N_1 is a *stationary* and N_2 is an ordinary renewal process with again $X_2^{(i)}$ having d.f. G_i, $i=1,2$. Now ζ_1 is denoted by $\tilde{\zeta}_1$. If we denote the 'lifetime' of N_1 (i.e. the time elapsed since the last renewal of N_1, or, if there was no renewal, the time elapsed since $t=0$) at time T_k by R, then we have

$$\begin{aligned} &P_{\mathscr{F}_{T_k}}(M(t_n)=n+k) \\ &\quad = I_{\{T_k=R\}}\tilde{Q}^{R,0}(M(t_n-T_k)=n)+I_{\{T_k\neq R\}}Q^{R,0}(M(t_n-T_k)=n). \end{aligned} \tag{5.2}$$

Let $(P^{r,0},\zeta)$ be $(\tilde{Q}^{r,0},\tilde{\zeta}_1)$ or $(Q^{r,0},\zeta_1^*)$. Then, applying Lemma 4.3, we have in either case

$$\begin{aligned} P^{r,0}(M(t)=n) &= \sum_{n_1=n}^{\infty} P^{r,0}(N_1(t_n)=n_1)P^{r,0}(N_2(t)=n_1-n) \\ &= \mathrm{O}(n^{-1/2}) \sum_{n_1=n}^{\infty} P^{r,0}(N_2(t)=n_1-n)=\mathrm{O}(n^{-1/2}), \end{aligned} \tag{5.3}$$

uniformly in $0\leq r<\zeta_1^*$ or $0\leq r<\tilde{\zeta}_1$, respectively and t. Hence, combining (5.1)–(5.3),

$$P(M(t_n)=n+k,\ T_k\leq t_n)=P(T_k\leq t_n)\mathrm{O}(n^{-1/2}), \tag{5.4}$$

where $\mathrm{O}(n^{-1/2})$ is independent of x and k. In particular,

$$\begin{aligned} \lim_{n\to\infty}\sqrt{n}\sum_{k=-\infty}^{-1} P(M(t_n)=n+k,\ T_k\leq t_n) &\leq \lim_{n\to\infty}\sum_{k=-\infty}^{-1} P(T_k\leq t_n)C \\ &= -CE\inf_{t\geq 0} M(t) \end{aligned} \tag{5.5}$$

by the monotone convergence theorem and the definition of T_k. Since $E\inf_{t\geq 0}M(t)$ is finite by Remark 3.1, Lemma 5.1 follows from (5.5). □

Lemma 5.2. *For fixed* $k\in\mathbb{Z}_-$,

$$\sup_{|x|\leq\log n}\left|P(M(t_n)=n+k,\ T_k\leq t_n)-\frac{\varphi(x)}{\gamma\sqrt{n\alpha}}P(T_k<\infty)\right|=\mathrm{o}(n^{-1/2}).$$

Proof. Let $\mathscr{F}_{T_k}$, $\tilde{Q}^{r,0}$, $Q^{r,0}$, $\tilde{\zeta}_1$ and ζ_1^* be defined as in the proof of Lemma 5.1. Fix $y>0$ and $\delta>0$. For n large enough

$$\begin{aligned} &P(M(t_n)=n+k,\ T_k\leq y) \\ &\quad \leq P(M(t_n)=n+k,\ T_k\leq t_n) \\ &\quad = P(M(t_n)=n+k,\ T_k\leq y)+P(M(t_n)=n+k,\ y<T_k\leq t_n). \end{aligned} \tag{5.6}$$

Using Lemmas 4.3 and 4.2 we have for $T_k \leqslant y$, similarly to (5.2) and (5.3),

$$P_{\mathscr{F}_{T_k}}(M(t_n)=n+k)=(I_{\{T_k=R,R\leqslant\tilde{\zeta}_1-\delta\}}+I_{\{T_k\neq R,R\leqslant\zeta_1^*-\delta\}})\frac{\varphi(x)}{\gamma\sqrt{n\alpha}}+\mathrm{o}_\delta(n^{-1/2})$$

$$+(I_{\{T_k=R,\tilde{\zeta}_1-\delta<R<\tilde{\zeta}_1\}}+I_{\{T_k\neq R,\zeta_1^*-\delta<R<\zeta_1^*\}})\mathrm{O}(n^{-1/2})$$

$$=f_R(\delta)\frac{\varphi(x)}{\gamma\sqrt{n\alpha}}+\mathrm{o}_\delta(n^{-1/2})+g_R(\delta)\mathrm{O}(n^{-1/2}),$$

where $f_R(\delta)=1$ and $g_R(\delta)=0$ if δ is small enough (depending on R) and $f_R(\delta), g_R(\delta)\in\{0,1\}$. Hence,

$$P(M(t_n)=n+k,\ T_k\leqslant y)$$

$$=EI_{\{T_k\leqslant y\}}P_{\mathscr{F}_{T_k}}(M(t_n)=n+k)$$

$$=EI_{\{T_k\leqslant y\}}\left\{\frac{\varphi(x)}{\gamma\sqrt{n\alpha}}f_R(\delta)+\mathrm{o}_\delta(n^{-1/2})+g_R(\delta)\mathrm{O}(n^{-1/2})\right\}$$

uniformly for $|x|\leqslant\log n$. Therefore

$$\lim_{y\to\infty}\lim_{\delta\downarrow 0}\lim_{n\to\infty}\left[\sqrt{n}\,P(M(t_n)=n+k,\ T_k\leqslant y)-\frac{\varphi(x)}{\gamma\sqrt{\alpha}}P(T_k\leqslant y)\right]$$

$$=\lim_{y\to\infty}\lim_{\delta\downarrow 0}\lim_{n\to\infty}EI_{\{T_k\leqslant y\}}\left\{\frac{\varphi(x)}{\gamma\sqrt{\alpha}}\{f_R(\delta)-1\}+\mathrm{o}_\delta(1)+g_R(\delta)\mathrm{O}(1)\right\}=0. \tag{5.7}$$

(Note that in taking the limit w.r.t. δ we apply the dominated convergence theorem.) Moreover, since, similarly to (5.4),

$$P(M(t_n)=n+k,\ y<T_k\leqslant t_n)\leqslant P(y<T_k\leqslant t_n)\mathrm{O}(n^{-1/2}),$$

where $\mathrm{O}(n^{-1/2})$ does not depend on y, we have

$$\lim_{y\to\infty}\lim_{n\to\infty}\sqrt{n}\,P(M(t_n)=n+k,\ y<T_k\leqslant t_n)=0. \tag{5.8}$$

Combination of (5.6)–(5.8) and $\lim_{y\to\infty}P(T_k\leqslant y)=P(T_k<\infty)$ now yields the result. □

Corollary 5.1. *Let M be the difference process of two independent stationary renewal processes and let T_k be defined as in Section 2, then*

$$\sup_{|x|\leqslant\log n}\left|\sum_{k=-\infty}^{-1}P(M(t_n)=n+k,\ T_k\leqslant t_n)+\frac{\varphi(x)}{\gamma\sqrt{n\alpha}}E\inf_{t\geqslant 0}M(t)\right|$$

$$=\mathrm{o}(n^{-1/2})\quad \textit{as } n\to\infty.$$

Proof. For every fixed $A \in \mathbb{N}$,

$$\sum_{k=-\infty}^{-1} P(M(t_n) = n+k,\ T_k \leqslant t_n)$$

$$= \sum_{k=-\infty}^{-A-1} P(M(t_n) = n+k,\ T_k \leqslant t_n) + \sum_{k=-A}^{-1} P(M(t_n) = n+k,\ T_k \leqslant t_n).$$

The result now follows from Lemmas 5.1 and 5.2. Note also that

$$\sum_{k=-\infty}^{-1} P(T_k < \infty) = -E \inf_{t \geqslant 0} M(t). \qquad \square$$

We now prove the main result, Theorem 3.1.

Proof of Theorem 3.1. Since $E\ \inf_{t\geqslant 0} M(t)$ is well defined by Remark 3.1, we have

$$\sup_{|x| \leqslant \log n} \left| P\left(\frac{\tau_n - n\alpha}{\gamma\alpha^{3/2}\sqrt{n}} \leqslant x\right) - \Phi(x) \right.$$
$$\left. - \frac{\varphi(x)}{\sqrt{n}} \left\{ p(1-x^2) + \frac{1}{2\gamma\sqrt{\alpha}} - \frac{\gamma\sqrt{\alpha}}{2} - \frac{1}{\gamma\sqrt{\alpha}} E \inf_{t\geqslant 0} M(t) \right\} \right|$$
$$= o(n^{-1/2}) \quad \text{as } n \to \infty,$$

by (3.2), Corollary 4.2, (3.3) and Corollary 5.1. Uniformity for all $x \in \mathbb{R}$ follows by the monotonicity of d.f.'s. $\square$

References

F.J. Anscombe, Sequential estimation, J. Roy. Statist. Soc. Ser. B 15 (1953) 1–21.

L.E. Baum and M. Katz, Convergence rates in the law of large numbers, Trans. Amer. Math. Soc. 120 (1965) 108–123.

D.R. Cox, Renewal Theory (Methuen, London, 1962).

W. Feller, An Introduction to Probability Theory and its Applications, Vol. II (Wiley, New York, 1971).

J. Kiefer and J. Wolfowitz, On the characteristics of the general queueing process, with applications to random walk, Ann. Math. Statist. 27 (1956) 108–123.

D.P. Kroese and W.C.M. Kallenberg, Approximations to the lifetime distribution of k-out-of-n systems with cold standby, Math. Oper. Res. 14(3) (1989) 485–501.

D.P. Kroese, The difference of two renewal processes: level crossing and the infimum, to appear in: Comm. Statist. Stochastic Models (1992).

S.P. Lalley, Limit theorems for first passage times in linear and non-linear renewal theory, Adv. in Appl. Probab. 16 (1984) 766–803.

D. Siegmund, Sequential Analysis: Tests and Confidence Intervals (Springer, New York, 1985).

M. Woodroofe, Non Linear Renewal Theory in Sequential Analysis (SIAM, Pennsylvania, PA, 1982).

M. Woodroofe and R. Keener, Asymptotic expansions in boundary crossing problems, Ann. Probab. 15 (1987) 102–114.