Voting Strategies, Power and Proof: Mathematics and Politics

Ken Smith

Kenneth Joseph Arrow
Born 1921

Awarded a half share in the
Nobel Prize for Economics in 1972
"for pioneering contributions to economic
and welfare theories"

Kenneth Arrow: 'A difficulty in the concept of social welfare', *Journal of Political Economy*, vol. 58 (1950), pp. 328–346.

For simplicity consider an electorate with four candidates, labelled A, B, C and X, and eleven voters. The voters have the following orders of preference for the candidates:

Voter	Preference			
Gerry	X	C	A	B
Harry	X	C	B	A
Irene	X	C	B	A
Joe	A	X	C	B
Karen	A	X	C	B
Lionel	A	C	X	B
Mary	A	C	B	X
Nigel	B	X	C	A
Olive	B	C	X	A
Peter	B	C	A	X
Queenie	C	B	X	A

Which candidate is elected depends on the voting system used.

If one Queensland slogan “Just vote 1!” were to become law then A would win with 4 voters placing her first.

Voter	Preference			
Gerry	X	C	A	B
Harry	X	C	B	A
Irene	X	C	B	A
Joe	A	X	C	B
Karen	A	X	C	B
Lionel	A	C	X	B
Mary	A	C	B	X
Nigel	B	X	C	A
Olive	B	C	X	A
Peter	B	C	A	X
Queenie	C	B	X	A

If full preferential voting (as applied in Federal elections) is used B is elected:

C is eliminated with the lowest total of first preferences;

then X is eliminated (3 first preferences while A and B have 4 each at this stage);

and then B defeats A by 6 to 5.

Voter	Preference			
Gerry	X	C	A	B
Harry	X	C	B	A
Irene	X	C	B	A
Joe	A	X	C	B
Karen	A	X	C	B
Lionel	A	C	X	B
Mary	A	C	B	X
Nigel	B	X	C	A
Olive	B	C	X	A
Peter	B	C	A	X
Queenie	C	B	X	A

A third form of voting applies weights to the preferences, since a first preference should carry more weight than a fourth preference.

One suggested way of weighting preferences is known as "Borda weighting": for 4 candidates the weights are 3, 2, 1 and 0 for 1st, 2nd, 3rd and 4th preferences respectively.

Under this system candidate C wins, with total weighted vote value of 20, compared to 18 for X and 14 each for A and B.

Voter	Preference			
Gerry	X	C	A	B
Harry	X	C	B	A
Irene	X	C	B	A
Joe	A	X	C	B
Karen	A	X	C	B
Lionel	A	C	X	B
Mary	A	C	B	X
Nigel	B	X	C	A
Olive	B	C	X	A
Peter	B	C	A	X
Queenie	C	B	X	A

Yet a fourth form of voting is pairwise comparison of the candidates. Thus in this case if we compare A and B, voters 2, 3, 8, 9, 10 and 11 (a majority of 6 out of 11) prefer B to A. Thus A is eliminated and we proceed to compare two of the remaining three.

A defect of this system is that the outcome can (and usually does) depend on the order in which comparisons are made. For the voting pattern here, however, X always wins, since different groups of 6 prefer him to each of the others.

Summary

4 candidates, 11 voters,
4 different voting systems:

1. "Just vote 1!", or "first past the post":
   Candidate A wins.

2. Full preferential voting:
   Candidate B wins.

3. Borda weighting of preferences:
   Candidate C wins.

4. Pairwise comparison:
   Candidate X wins.

Since full preferential voting is used in Federal elections in Australia, let us look further into it. Consider the following voting pattern for three candidates A, B and C in two parts (East and West for convenience) of an electorate.

Order of Preference	Votes cast		
	East district	West district	Total
ABC	160	257	417
ACB	0	82	82
BAC	143	0	143
BCA	0	357	357
CAB	0	285	285
CBA	285	39	324
Totals	588	1020	1608

The first preference votes for each candidate are:

Candidate	East district	West district	Total
A	160	339	499
B	143	357	500
C	285	324	609
Totals	588	1020	1608

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Candidate	East district	West district	Total
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Totals	588	1020	1608

We now proceed through the counting procedure.

In the East district B is eliminated. Since all B's second preferences go to A, she now defeats C by 303 votes to 285.

In the West district C is eliminated first, and then A again wins, this time by 624 votes to 396.

But if we look at the whole electorate A is the first to be eliminated, and B wins by 917 votes to 691.

What is going on?

Two extra voting papers for the East district: both voters preferred A to B to C.

Order of Preference	Votes cast		
	East district	West district	Total
ABC	162	257	419
ACB	0	82	82
BAC	143	0	143
BCA	0	357	357
CAB	0	285	285
CBA	285	39	324
Totals	590	1020	1610

Now the first preference votes for each candidate are:

Candidate	East district	West district	Total
A	162	339	501
B	143	357	500
C	285	324	609
Totals	590	1020	1610

Now for the electorate as a whole B is eliminated first, and C defeats A by 966 votes to 644. Two extra votes putting C last turn him from a loser into a winner!

Two electors in the West district who voted ABC really meant to vote CAB.

Order of Preference	Votes cast		
	East district	West district	Total
ABC	162	255	417
ACB	0	82	82
BAC	143	0	143
BCA	0	357	357
CAB	0	287	287
CBA	285	39	324
Totals	590	1020	1610

Now the first preference votes for each candidate are:

Candidate	East district	West district	Total
A	162	337	499
B	143	357	500
C	285	326	611
Totals	590	1020	1610

A is eliminated first, and B beats C by by 915 votes to 693. Two voters changing their minds about C, and putting him first instead of last, turn him from a winner into a loser!

This unexpected behaviour is not restricted to full preferential voting. All other voting systems in use have their defects.

In Australia proportional representation in Senate elections has resulted in one person, or a small group, having an influence out of all proportion to their support in the community.

"First past the post" as used in most other places also has difficulties. In elections for the President of France a modified form of "first past the post" is used: the two candidates who gain most votes in a preliminary ballot are submitted to the voters for a final ballot. In the last such election those voters on the left found themselves, much against their will, voting for a right-wing candidate, so that an even more right-wing candidate wouldn't gain office.

Since none of the available voting systems are satisfactory, what is needed is some system which will, at least to some extent, reflect “the will of the people”.

To be more precise, it should reflect the will of “rational” members of the community, if we can come up with some way of defining this.

Unfortunately, as Kenneth Arrow showed in a classic paper in 1950, our search for a rational voting system which reflects a bare minimum of what we might regard as a “democratic” system, must remain unfulfilled.

This work was the main contribution which resulted in the award of a share in the 1972 Nobel Prize for Economics. The award of a Nobel Prize is normally for advancing our knowledge in some area: it must be a rare occurrence for the prize to be awarded for showing that something was impossible.

Arrow started with some simple assumptions about how voters individual preferences could be combined into an overall preference list for the electorate — or how consumers individual preferences for canned drinks could be combined into an overall preference for refreshment.

Mathematicians would call these axioms.

1. The voting system does not constrain the voters' choice: by suitable listing of their preferences the voters can force any outcome.

2. There are at least three choices available.

3. Voters are rational, in a minimal sense. If a voter prefers candidate X to candidate Y, and prefers candidate Y to candidate Z, then surely she or he will prefer X to Z. This should also be true for the overall preference list for the electorate.

Then there are two assumptions about the relationship between the individual preference lists and the overall preference list.

4. If one candidate is raised in the preference list of one (or more) voter(s), then that candidate should not be lower down in the overall preference list.

5. Whether candidate X is listed above or below Y in the overall preference list depends only on the relative preferences of the voters between X and Y: preferences for other candidates should not affect how X and Y are listed.

Apart from the requirement for at least three candidates, each of these conditions can be replaced by a different plausible condition without affecting the conclusion. However the proofs for some of these become quite lengthy and involved.

Some Notation

In Arrow's words "It will be found convenient to represent preference by a notation not customarily employed in economics, though familiar in mathematics . . . "

The assumed transitive nature of preference (if X is preferred to Y, and Y to Z, then surely X is preferred to Z) indicates that we have some sort of order relation between the candidates.

Each voter will have an individual sequence of preferences, and will prefer the candidates in some order.

The problem is to translate the informal statements shown earlier into a precise mathematical form, both for individuals and the electorate as a whole.

Some Notation — continued

We consider a group V of n voters:

$$V = \{v_1, v_2, v_3, \ldots v_n\},$$

and a group H of political hopefuls,

$$H = \{a, b, c, \ldots\}.$$

Each voter will arrange the candidates in some preferred order. This is equivalent to setting up a relation between candidates.

For each v_i a relationship R_i is defined between each pair of elements in H:

1. For each pair (x, y) either xR_iy or yR_ix holds.

2. This relation is reflexive and transitive, but not necessarily symmetric:

   xR_ix;

   xR_iy and yR_iz implies xR_iz;

   If xR_iy then yR_ix may or may not hold.

Such a relation is called a *weak order* on H, and there will be one weak order for each elector.

Preference or indifference between candidates is easily defined in terms of the relation R_i:

Preference :

Voter v_i prefers x to y if xR_iy but not yR_ix.
This will normally be denoted xP_iy.

Indifference :

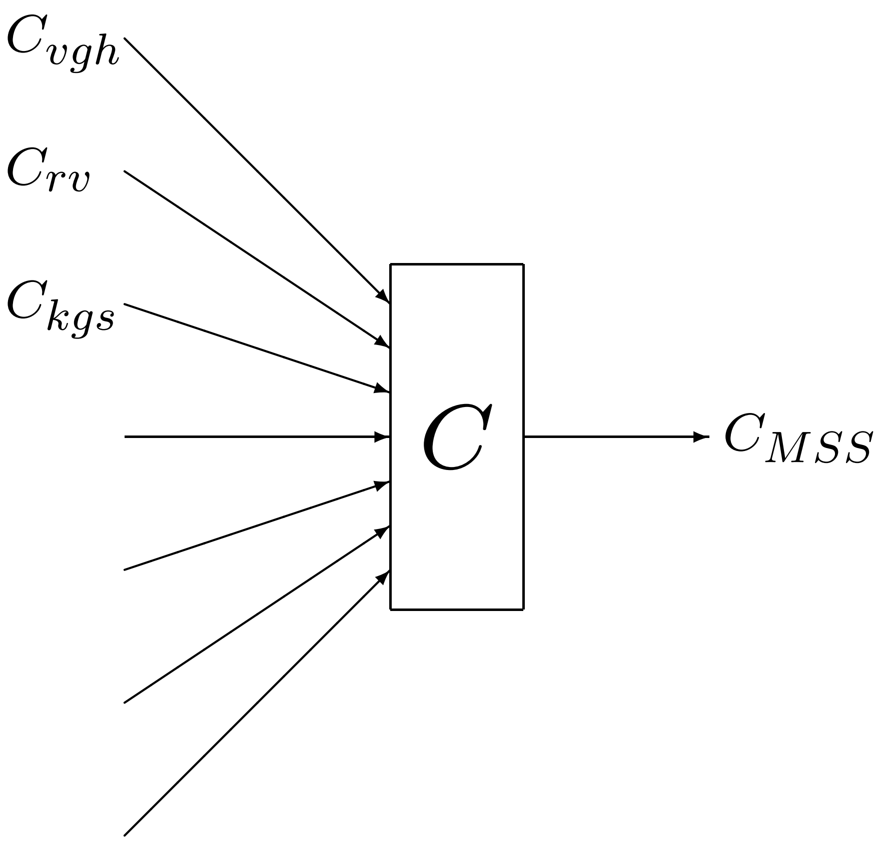
Voter v_i has no preference between x and y (is indifferent to their relative merits) if xR_iy and yR_ix.
This will normally be denoted xI_iy.

Relations without subscripts, xRy, xPy and xIy will refer to the order for the electorate, as determined by the voting procedure.

The problem is to combine the individual order relations to find a global order. If we label the choice (i.e., the collected relations) of voter v_i as C_i, then the problem is finding a function C for the electorate which depends on the collections:

$$C = C(C_1, C_2, C_3, \ldots C_n).$$

Example: Mathematics Students' Society.



Let us collect the axioms to be used.

Axiom 1 (Freedom): The voting system does not impose constraints. Any possible output order can be achieved by some combination of inputs.

Axiom 2: There are at least three candidates.

Axiom 3 (Rationality): xP_iy and yP_iz imply xP_iz. Also xPy and yPz imply xPz.

Axiom 4 (Independence of Irrelevant Alternatives): The final choice between x and y is determined by the relative placing of these on the voters' lists: the position of other candidates is irrelevant.

Axiom 5 (Monotonicity, or "welfare" not "ill fare"): If one voter changes from yP_ix to xP_iy then x should not be moved lower in the final list.

The importance of Arrow's work can be summed up in two questions and answers.

Question 1: Is there any system which satisfies Axioms 1–5?

Answer 1: Yes. Appoint a dictator.

A dictator is someone whose choice is always followed, irrespective of any preferences the remainder of the community may have. In this case all inputs to the function C other than that from the dictator are ignored.

Question 2: Is there any other solution?

Answer 2: No.

The proof of answer 2 is widely known as "Arrow's Impossibility Theorem".

We will prove some lemmas and then the theorem. We will restrict ourselves to the special case where only preference relations are allowed in the individual preference lists: the extension to including indifference relations will be mentioned later.

Since we have already mentioned a dictator, we will introduce some terminology relating to dictatorship.

Definition: Let S be a subset of V. We say that "S can force x over y" to mean that we get xPy whenever everyone in S has xP_iy.

To show that some subset S has this property we need only construct a single collection of individual preference lists with the following properties:
1. Everyone in S has xP_iy;
2. Everyone not in S has yP_jx;
3. The resulting list has xPy.

Such a set S is called a *forcing set* or a *dictating set* for x and y.

The first lemma shows that the whole electorate V is a dictating set. This is uncontroversial: if everyone votes the same way that should be the final result. We will (eventually) prove the other extreme: there a single individual who forms a dictating set.

In some expositions of Arrow's results Axiom 1 is replaced by the weaker axiom that if everyone has an identical preference list then this is the overall preference list.

Lemma 1 : *If xP_iy for all i in V then xPy.*

Proof:
(a) By Axiom 1 there is some combination of choices C_i which will result in xPy.

(b) Let all voters move x to the top of their preference list. Then by Axiom 5 we will still have xPy.

(c) By Axiom 4 (Independence of alternatives) the result follows. □

The next Lemma shows that if individual preference lists are not permitted to include indifference relations (as in Federal elections) then the output list will not include any indifference relations.

Lemma 2 : *If there are at least three candidates, then if there are no indifference relations in the individual choices there will be no indifference relations in the final choice.*

Proof: Suppose, on the contrary, that for some x and y we have xIy. Then some of the voters will have xP_iy: collect these into set A. The remaining voters (in set B for convenience) will have yP_jx.

We now use Axiom 2 to introduce candidate z, and consider two different positions for z.

(a) Let all i in A have a preference list in which xP_izP_iy, and all j in B have a preference list in which zP_jyP_jx. From these we have zP_ky for all k in V, and so, from Lemma 1, zPy. Thus we have $zPyIx$, or $zPxIy$. Now from Axiom 4, as far as x and z are concerned, the position of y is irrelevant, so we have zPx.

Proof of Lemma 2 (continued):

(b) Let all i in A have a preference list in which xP_iyP_iz, and all j in B have a preference list in which yP_jzP_jx. For these we have yP_kz for all k in V, and so, from Lemma 1, yPz. Thus we have $xIyPz$. Now from Axiom 4, as far as x and z are concerned, the position of y is irrelevant, so we have xPz. This contradicts the result zPx from part (a), and so our assumption that xIy must be false. $\square$

The next three Lemmas relate to the remarkable power of a forcing set. This was initially defined in terms of two specific candidates, say x and y. These Lemmas show that if there is a set which can dictate the overall relationship between x and y, then it can dictate the overall relationship between any two candidates.

Lemma 3 : *Suppose S forces x over y, and z is an alternative distinct from x and y. Now split S into two sets T and U (either of which may be the empty set) so that each element of S is in exactly one of T and U. Then either T forces x over z, or U forces z over y.*

(Informal statement of Lemma: If S has the power to force x high and y low, then either T inherits the power to force x high or U inherits the power to force y low.)

Proof: Consider what happens when we have the following sequence of individual lists.
(a) xP_iyP_iz for all i in T;
(b) zP_jxP_jy for all j in U;
(c) yP_kzP_kx for all k in the remaining voters in V.
Everyone in T and U, and thus everyone in S, has x over y, so xPy.

This means that we cannot have both yPz and zPx, since transitivity would give yPx.

Thus we must have at least one of xPz or zPy, and these two cases will be treated separately.

Proof of Lemma 3 continued:

Case 1: we have xPz.
From (a) above everyone in T has xP_iz, while from (b) and (c) everyone not in T has zP_jx.
Thus T inherits the power to force x over an arbitrary element z, and the Lemma is proved in this case.

Case 2: we have zPy.
An analogous argument — left as an exercise — shows that in this case U inherits the power to force an arbitrary element z over y. $\square$

Historical note: The preference relations in (a), (b) and (c) above were used in the 18th century by the Marquis de Condorcet to indicate a paradox about allocating preferences.

Lemma 4 : *Suppose that S forces x over y, and that z is distinct from both x and y (in accordance with Axiom 2). Then S forces x over z and S forces z over y.*

(Informal statement of Lemma: If S can force x over y, or, equivalently, can force y under x, then it can force x over anything, and can force y under anything, that is, can force anything over y.)

Proof: Consider first the special case of Lemma 3 where T is the whole set S and U is the empty set, $\emptyset$. Then either T forces x over z or $\emptyset$ forces z over y. The second of these means that zPy even if this is opposed by all voters, and thus contradicts Axiom 1, so the first part of the lemma is proved.

In an analogous way by taking $T = \emptyset$ the second part of the Lemma can be proved. □

Lemma 5 : *If S forces x over y, then S forces y over x.*

(Informal statement of Lemma: The forcing relation is symmetric.)

Proof: Choose an element z of H distinct from both x and y: this is possible by Axiom 2.

Since S forces x over y, by Lemma 4 S forces x over anything, in particular it forces x over z.

By Lemma 4 again we have that S forces z under anything: in particular it forces z under y.

This is the same as saying that S forces y over z.

By using Lemma 4 one more time we see that S forces y over anything, and so it forces y over x as desired. $\square$

Lemma 6 : *If there are two distinct elements x and y in H such that S can force x over y, then S is a dictating set.*

(Informal statement of Lemma: If S has a little local power, then S has complete global power.)

Proof: Let z and w be two arbitrary elements of H. It will be sufficient to show that S can force z over w, since Lemma 5 then ensures that S can force w over z.

Lemma 5 ensures that S can force y over x, and then Lemma 4 ensures that S can force x over or under anything and can also force y over or under anything.

Case 1: $x = w$
Since we know that S can force x under anything it can force x under z, i.e., it can force z over w.

Case 1: $x \neq w$
Since S can force x over anything it can force x over w. Hence it can force w under anything (by Lemma 4) and so can force w under z, i.e., it can force z over w. $\Box$

Lemma 7 : *Suppose that S is a dictating set. Split it into two subsets T and U such that each element of S is in exactly one of T and U. Then either T or U is a dictating set.*

Proof:
Choose three distinct elements x, y and z in H.

S can force x over y, so Lemma 3 ensures that either T can force x over z (in which case T is a dictating set by Lemma 6), or U can force z over y (in which case U is a dictating set by Lemma 6 again). □

Theorem 1 *There is a dictator.*

Proof: Apply Lemma 7 to the set V which we showed was a dictating set. Then apply Lemma 7 to either T or U, whichever is the dictating set. Then apply Lemma 7 to . . . □

Additional material

The restriction that the individual preference list may not contain any indifference relations can be removed. Some of the proofs become longer, and the definition of a dictator must be slightly relaxed to a "weak" dictatorship.

If subscript d is used for the weak dictator, then xP_dy implies that xPy, but zI_dw can lead to any of zIw, zPw or wPz, each of which is compatible with a dictator who doesn't care if people insist on making a choice where he doesn't.

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