Exact solutions of nonlinear bosonic models

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Outline

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Back to basics - QISM

Central to the Quantum Inverse Scattering Method (QISM) is the construction of a family of commuting matrices known as **transfer matrices** (Korepin 1993).

That is, for operator $t(u),\ u \in \mathbb{C}$ acting on some vector space V representing the Hilbert space of physical states we have

$$[t(u), t(v)] = 0, \quad \forall u, v \in \mathbb{C}.$$

Consequences:

1. $t(u)$ can be diagonalised independent of u.
(ie evecs of $t(u)$ are independent of u.)
2. Taking the series expansion

$$t(u) = \sum_{k=-\infty}^{\infty} t_k u^k \text{ we have } [t_k, t_j] = 0, \quad \forall k, j.$$

- So for any Hamiltonian that can be expressed as a function of the operators t_k, each t_k represents a **constant of the motion** as it commutes with the Hamiltonian.
- When the number of independent conserved quantities equals the number of degrees of freedom of the system, the model is **integrable**.

In the theory of exactly solvable systems we begin with the *R*-matrix (invertible) which defines the **Yang-Baxter Algebra**, Y, generated by the monodromy matrix $T(u)$:

$$R_{12}(u-v)T_{13}(u)T_{23}(v) = T_{23}(v)T_{13}(u)R_{12}(u-v)$$

which acts in $End(V \otimes V) \otimes Y$.
The Yang-Baxter equation provides the associativity of Y:

$$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v).$$

Taking a realisation of Y, we obtain the L-operator acting in $\mathrm{End}(V \otimes W)$ which satisfies

$$R_{12}(u-v)L_{13}(u)L_{23}(v) = L_{23}(v)L_{13}(u)R_{12}(u-v).$$

If $L(u)$ is an L-operator, then so is $L(u+\alpha)$ for any α.

We define the **transfer matrix** through

$$t(u) = \pi(\mathrm{Tr}_0(T(u)))$$

and obtain $[t(u), t(v)] = 0, \quad \forall u, v \in \mathbb{C}$.

The algebra Y has the structure of a **bi-algebra**, so that we have the homomorphisms

$$\Delta : Y \to Y \otimes Y, \; \epsilon : Y \to \mathbb{C}$$

which are defined by

$$\begin{aligned} \Delta(T^j_k(u)) &= \sum_{l=1}^{n} T^l_k(u) \otimes T^j_l(u) \\ \epsilon(T^j_k(u)) &= \delta^j_k \end{aligned}$$

for $T(u) = \sum_{j,k=1}^{n} e^j_k \otimes T^j_k(u)$, with elementary matrices e^j_k.

The **coproduct** will be used to build a tensor product of representations of Y.

- The R-matrices that we work with depend on the spectral parameter u and an arbitrary complex parameter η.
  Eg. $R(u) = \frac{1}{u+\eta}(uI \otimes I + \eta P)$,
  where P is the permutation operator.

- These R-matrices have the property

$$\lim_{\eta \to 0} R(u) = I \otimes I$$

  which is known as the **quasi-classical property**.

- We can expand the R-matrix and monodromy matrix $T(u)$ in η to obtain the Gaudin algebra from Y.

Algebraic Bethe Ansatz

We apply the algebraic Bethe ansatz approach to diagonalise the transfer matrix. Suppose we represent the monodromy matrix as

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} .$$

The first step is to find a suitable vacuum state ξ which has the properties

$$\begin{aligned} A(u)|\xi\rangle &= a(u)|\xi\rangle \\ B(u)|\xi\rangle &= 0 \\ C(u)|\xi\rangle &\neq 0 \\ D(u)|\xi\rangle &= d(u)|\xi\rangle \end{aligned}$$

where $a(u),\ d(u)$ are scalar functions.

Then we choose the Bethe state to be a product of the operators $C(u_k)$ and determine the action of the transfer matrix on this state.

$$
\begin{aligned}
|\Omega\rangle &= \prod_{k=1}^{M} C(u_k)|\xi\rangle, \\
t(u)|\Omega\rangle &= [A(u) + D(u)]|\Omega\rangle.
\end{aligned}
$$

The action leads to the requirement that certain "unwanted terms" cancel, this leads to the Bethe ansatz equations (BAE).

Thus the Bethe state is an eigenstate of the transfer matrix and we are able to obtain the eigenvalue whenever the BAE are satisfied.

Hamiltonian

We begin with the Hamiltonian

$$H = \epsilon(S^+ b + S^- b^\dagger) + \gamma S^z + (S^z)^2 - C/4$$

with

- $su(2)$ spin operators
  $[S^z, S^\pm] = \pm S^\pm, \; [S^+, S^-] = 2S^z$
- $C = S^- S^+ + S^+ S^- + 2(S^z)^2$, the Casimir invariant
- molecular boson operators $b, b^\dagger$ which obey
  $[b, b^\dagger] = 1, \quad M = b^\dagger b$

Boundary Quantum Inverse Scattering Method

R-matrix $\rightarrow$ *L-operator* $\rightarrow$ *Monodromy matrix* $\rightarrow$ *transfer matrix* $\rightarrow$ *Hamiltonian*

Take a rational solution of the Yang-Baxter equation and apply the boundary QISM (Sklyanin 1988).

We construct a family of commuting **transfer matrices**

$$[t(u), t(v)] = 0, \quad \forall u, v \in \mathbb{C}.$$

Taking the series expansion $t(u) = \sum_{k=-\infty}^{\infty} t_k u^k$ we have

$[t_k, t_j] = 0, \quad \forall k, j$ *constants of the motion* which we use to construct the Hamiltonian so that resulting model is **integrable**.

Yang-Baxter equation:

$$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v)$$

$su(2)$ invariant R-matrix solution

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$b(u) = u/(u+\eta), \quad c(u) = \eta/(u+\eta),$
η is an arbitrary complex parameter.

The L-**operator** is given by the realisation

$$L(u) \quad = \quad I + \frac{\eta}{u} S$$

for

$$S = \begin{pmatrix} S^z & S^- \\ S^+ & S^z \end{pmatrix} .$$

The operator $\hat{L}(u) = I - \dfrac{\eta}{u-\eta} S$, satisfies

$$L(u)\hat{L}(u) = \left(1 - \frac{3\eta^2}{4u(u-\eta)}\right) I$$

and the L-operator generates the Yang-Baxter algebra
$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u-v)$.

We will also use another realisation of the Yang–Baxter algebra written in terms of the canonical boson operators b, $b^\dagger$ (Links *et al* JPA, 2003)

$$J(u) = \begin{pmatrix} 1 + \eta u + \eta^2(M+1) & \eta b \\ \eta b^\dagger & 1 \end{pmatrix}$$

with

$$\hat{J}(u) = \begin{pmatrix} 1 & -\eta b \\ -\eta b^\dagger & 1 + \eta u + \eta^2 M \end{pmatrix},$$

where $J(u)\hat{J}(u) = (1+\eta u)I$.

So the operator $J(u)$ obeys
$R_{12}(u-v)J_1(u)J_2(v) = J_2(v)J_1(u)R_{12}(u-v)$.

We construct a **doubled Monodromy matrix** as follows

$$T(u) \;=\; L(u-\epsilon)J(u)K_-(u)\hat{J}(-u)\hat{L}(-\epsilon-u)$$

where we introduce the boundary K-matrices given by (Hikami 1995)

$$K_- \;=\; \begin{pmatrix} u+\alpha_1 & 0 \\ 0 & \alpha_1-u \end{pmatrix}, \quad K_+ = \begin{pmatrix} u+\alpha_2+\eta & 0 \\ 0 & \alpha_2-u-\eta \end{pmatrix}.$$

These boundary K-matrices satisfy Sklyanin's reflection equations

$$\begin{aligned} &R_{12}(u-v)(K_-(u)\otimes I)R_{12}(u+v)(I\otimes K_-(u)) \\ &\qquad = (K_-(v)\otimes I)R_{12}(u+v)(K_-(u)\otimes I)R_{12}(u-v) \end{aligned}$$

$$\begin{aligned} &R_{12}(-u+v)(K_+^t(u)\otimes I)R_{12}(-u-v-2\eta)(I\otimes K_+^t(u)) \\ &\qquad = (K_+^t(v)\otimes I)R_{12}(-u-v-2\eta) \\ &\qquad \times (K_+^t(u)\otimes I)R_{12}(-u+v) \end{aligned}$$

where $K_+^t(u)$ is the transpose of $K_+(u)$.

It follows that the doubled monodromy matrix satisfies the boundary YBE

$$R_{12}(u-v)(T(u)\otimes I)R_{12}(u+v)(I\otimes T(v)) \\ = (I\otimes T(v))R_{12}(u+v)(T(u)\otimes I)R_{12}(u-v).$$

The transfer matrix is given by

$$t(u) \quad = \quad \mathrm{Tr}_0[K_+(u)T(u)]$$

which through the doubled monodromy matrix can be shown to provide a family of commuting matrices:

$$[t(u),\, t(v)] = 0 \qquad \forall\, u, v \in \mathbb{C}.$$

Taking a series expansion of the transfer matrix leads to a set of mutually commuting operators,

$$\begin{aligned} t(u) &\approx u^2 t_{11} + u t_0 + u^0 t_1 + u^{-1} t_2 + u^{-2} t_3, \\ [t_k, t_j] &= 0, \quad \forall k, j. \end{aligned}$$

Using the Casimir invariant $C = S^- S^+ + S^+ S^- + 2(S^z)^2$
and setting $\alpha_1 = \alpha\eta,\ \alpha_2 = 0, \gamma = \alpha + \epsilon^2$,
the Hamiltonian is related to t_1 by

$$H = \lim_{\eta \to 0} \left(\frac{t_1}{4\eta^2} \right).$$

Algebraic Bethe ansatz

Eigenvalues of the transfer matrix - Sklyanin 1988, Hikami 1995.

Our ingredients:

- R-matrix
- spin L-operator $L(u)$
- bosonic L-operator $J(u)$
- the boundary K matrices
- doubled monodromy matrix $T(u)$.

The transfer matrix is given by

$$t(u) = Tr_0[K_+ T(u)].$$

Through the **boundary Yang-Baxter equation**

$$R_{12}(u-v)(T(u)\otimes I)R_{12}(u+v)(I\otimes T(v))$$
$$=(I\otimes T(v))R_{12}(u+v)(T(u)\otimes I)R_{12}(u-v)$$

we solve the eigenvalue problem

$$t(u)\Psi=\Lambda(u)\Psi$$

by the open boundary ABA method.
Represent the doubled monodromy matrix in terms of the operators

$$T(u)=\begin{pmatrix}A(u) & B(u)\\ C(u) & D(u)\end{pmatrix},$$

so that the transfer matrix is given by

$$t(u)\Psi=\left[(u+\eta+\alpha_2)A(u)+(\alpha_2-\eta-u)D(u)\right]\Psi.$$

For the diagonalization of the transfer matrix, it is necessary to introduce a change of variable, viz.

$$\hat{A}(u) = (2u+\eta)A(u) - \eta D(u).$$

Then the relevant relations arising from the algebra may be expressed as

$$\begin{aligned}
\hat{A}(u)C(v) &= \frac{(u-v+\eta)(u+v+2\eta)}{(u+v+\eta)(u-v)}C(v)\hat{A}(u) \\
&\quad -\frac{2\eta(u+\eta)}{(2v+\eta)(u-v)}C(u)\hat{A}(v) \\
&\quad +\frac{4v\eta(u+\eta)}{(2v+\eta)(u+v+\eta)}C(u)D(v) \\
D(u)C(v) &= \frac{(u+v)(u-v-\eta)}{(u+v+\eta)(u-v)}C(v)D(u) \\
&\quad +\frac{2v\eta}{(2v+\eta)(u-v)}C(u)D(v) \\
&\quad -\frac{\eta}{(u+v+\eta)(2v+\eta)}C(u)\hat{A}(v).
\end{aligned}$$

The transfer matrix now becomes

$$t(u) = \frac{1}{2u+\eta}\left[(u+\eta+\alpha_2)\hat{A}(u) + [2\alpha_2\eta + 2u(\alpha_2-\eta-u)]D(u)\right].$$

We choose the reference state to be a lowest weight state

$$|\Omega\rangle = |-I \otimes 0\rangle$$

such that for the construction of the eigenstates we have $B(u)|\Omega\rangle = 0$.

doubled Monodromy matrix

$$T(u) \;\; = \;\; L(u-\epsilon)J(u)K_-(u)\hat{J}(-u)\hat{L}(-\epsilon-u).$$

The action of the diagonal elements of the transfer matrix on this reference state are

$$\begin{aligned}\hat{A}(u)|\Omega\rangle &= 2u(u+\eta+\alpha_1)\alpha(u)\hat{\alpha}(-u)|\Omega\rangle \\ D(u)|\Omega\rangle &= (\alpha_1 - u)\delta(u)\hat{\delta}(-u)|\Omega\rangle,\end{aligned}$$

where

$$\begin{aligned}\alpha(u)|\Omega\rangle &= (1+\eta u+\eta^2)\left(\frac{u-\epsilon-\eta l}{(u-\epsilon)}\right)|\Omega\rangle, \\ \hat{\alpha}(-u)|\Omega\rangle &= \left(\frac{u+\eta+\epsilon-\eta l}{(u+\eta+\epsilon)}\right)|\Omega\rangle, \\ \delta(u)|\Omega\rangle &= \left(\frac{u-\epsilon+\eta l}{(u-\epsilon)}\right)|\Omega\rangle, \\ \hat{\delta}(-u)|\Omega\rangle &= (1-\eta u)\left(\frac{u+\eta+\epsilon+\eta l}{(u+\eta+\epsilon)}\right)|\Omega\rangle.\end{aligned}$$

The Bethe ansatz states are chosen to be a product of the creation operators acting on the reference state,

$$\Psi = \prod_{j=1}^{N} C(v_j)|\Omega\rangle.$$

So for the transfer matrix: $t(u)\Psi = \Lambda(u)\Psi +$ "unwanted terms", the co-efficient $\Lambda(u)$ is given by

$$\begin{aligned}
\Lambda(u) \quad = \quad & \frac{2u(u+\alpha_1+\eta)(u+\alpha_2+\eta)(1+\eta u+\eta^2)}{2u+\eta} \\
\times \quad & \prod_{j=1}^{N} \frac{(u-v_j+\eta)(u+v_j+2\eta)}{(u+v_j+\eta)(u-v_j)} \frac{(u-\epsilon-\eta l)(u+\epsilon+\eta(1-l))}{(u-\epsilon)(u+\epsilon+\eta)} \\
+ \quad & \frac{2(1-u\eta)(\alpha_1-u)[\alpha_2\eta+u(\alpha_2-\eta-u)]}{2u+\eta} \\
& \times \prod_{j=1}^{N} \frac{(u+v_j)(u-v_j-\eta)}{(u+v_j+\eta)(u-v_j)} \frac{(u-\epsilon+\eta l)(u+\epsilon+\eta(l+1))}{(u-\epsilon)(u+\epsilon+\eta)}.
\end{aligned}$$

For the cancellation of the "unwanted terms", the roots must satisfy the Bethe ansatz equations (BAE)

$$\frac{(1-\eta v_k)(\alpha_1 - v_k)[\alpha_2\eta + (\alpha_2 - \eta - v_k)v_k]}{v_k(v_k+\alpha_1+\eta)(v_k+\eta+\alpha_2)(1+\eta v_k+\eta^2)} \times \frac{(v_k+\epsilon+\eta(1+l))(v_k-\epsilon+\eta l)}{(v_k-\epsilon-\eta l)(v_k+\epsilon+\eta(1-l))} = \prod_{j\neq k}^{N} \frac{(v_k-v_j+\eta)(v_k+v_j+2\eta)}{(v_k+v_j)(v_k-v_j-\eta)}, \qquad k = 1 \ldots N.$$

Results in the quasi-classical limit

To obtain the energy corresponding to the Hamiltonian, we set $\alpha_1 = \alpha\eta,\ \alpha_2 = 0$ and expand the eigenvalue of the transfer matrix as

$$\Lambda(u) = \lambda_{11} u^2 + \lambda_0 u + \lambda_1 u^0 + \lambda_2 u^{-1} + \lambda_3 u^{-2}.$$

The resulting energy of the Hamiltonian arises from λ_1 and is given by

$$E = N(N + \alpha - 2l) - \frac{l}{2}(1 + 2\alpha - l + 2\epsilon^2) + \sum_{k=1}^{N} v_k^2,$$

subject to the BAE

$$-1 - \frac{2\alpha + 1}{2v_k^2} + \frac{2l}{v_k^2 - \epsilon^2} = \sum_{j \neq k, j=1}^{N} \frac{2}{v_k^2 - v_j^2}, \quad k = 1 \dots N.$$

Mapping to a Schrödinger equation

Setting $x_k = v_k^2,\ \epsilon^2 = \theta$ and $x_k \to x_k + \theta/2$, we have an extension of the general case for hyperbolic potentials presented in Dunning *et al* preprint 2008.

That is, we express the BAE as

$$A + \frac{B}{x_k + \theta/2} + \frac{C}{x_k - \theta/2} = \sum_{j \neq k} \frac{2}{x_k - x_j}$$

with the values $A = -1,\ B = -\alpha - \frac{1}{2},\ C = 2I$ and the corresponding energy given by

$$E = \sum_{k=1}^{N} x_k + DN + F$$

where $D = (N + \alpha - 2I),\ F = -\frac{I}{2}(1 + 2\alpha - I + 2\theta)$.

In the form of a Schrödinger equation we write (see Ulyanov 1992),

$$-\quad \frac{d^2\Psi(x)}{dx^2} + V(x)\Psi(x) = E\ \Psi(x)$$

where we find

$$\begin{aligned} V(x) \quad = \quad & N\left(-\frac{\theta}{2}\cosh(x) - \frac{1}{2}\right) - \frac{l}{2}(1 + 2\alpha - l + 2\theta) + \frac{\theta^2}{16}\sinh^2(x) \\ & + \frac{1}{4}\left(2l - \alpha + \frac{1}{2}\right)^2 - \frac{(4l+1)(4l+3)}{8(1+\cosh(x))} + \frac{\alpha(\alpha-1)}{2(\cosh(x)-1)} \\ & - \frac{\theta}{4}\left[2l + \alpha + \frac{1}{2} + (\alpha - 2l + \frac{1}{2})\cosh(x)\right] \end{aligned}$$

with the corresponding energy and wave function given by

$$E = \sum_{k=1}^{N} x_k + N(N + \alpha - 2l) - \frac{l}{2}(1 + 2\alpha - l + 2\theta)$$

and

$$\begin{aligned}\Psi(x) &= (1 - \cosh(x))^{\alpha/2}(1 + \cosh(x))^{-l-1/4} \exp\left(-\frac{\theta}{4}\cosh(x)\right) \\ &\quad \times \prod_{k=1}^{N}\left(\frac{\theta}{2}\cosh x + x_k\right)\end{aligned}$$

respectively.

Relation to other Models

We show that for two different realisations of the $su(2)$ operators, we are able to recover the models of G. Santos *et al* 2006 and M. Duncan *et al* 2007.

For $\{\alpha, \alpha^\dagger : \alpha = a, b\}$ canoncial boson operators with $n_\alpha = \alpha^\dagger \alpha$, we take the realisations of $su(2)$ operators, viz.

$$S^+ = a^\dagger b, \quad S^- = ab^\dagger, \quad S^z = \frac{1}{2}(n_a - n_b),$$

we obtain the Hamiltonian in G. Santos *et al* 2006

$$H = U_a N_a^2 + U_b N_b^2 + U_{ab} N_a N_b + \mu_a N_a + \mu_b N_b + \Omega(a^\dagger a^\dagger b + b^\dagger aa).$$

with the identifications $U_a = U_b = 1/8,\ U_{ab} = 1/8,\ \Omega = \epsilon,$
$\mu_a = -\gamma/2 - 1/4,\ \mu_b = \gamma/2 - 1/4.$
The Hamiltonian describes coupling between atomic and diatomic-molecular BECs and is a generalisation of the two-site Bose-Hubbard models studied in quantum tunneling between two single-mode BECs. (See Tonel *et al* 2005 amongst others.)

Alternatively, for $\{\alpha, \alpha^\dagger : \alpha = a, c\}$ canoncial boson operators with $n_\alpha = \alpha^\dagger \alpha$, we can choose the following realisations of $su(2)$ operators,

$$S^+ = a^\dagger c, \qquad S^- = ac^\dagger, \qquad S^z = \frac{1}{2}(n_a - n_c),$$

and we obtain the Hamiltonian of Duncan *et al* 2007

$$\begin{aligned} H &= U_{aa}N_a^2 + U_{bb}N_b^2 + U_{cc}N_c^2 + U_{ab}N_aN_b + U_{ac}N_aN_c + U_{bc}N_bN_c \\ &\quad + \mu_a N_a + \mu_b N_b + \mu_c N_c + \Omega(a^\dagger b^\dagger c + c^\dagger ba). \end{aligned}$$

with the identifications
$U_{aa} = U_{cc} = 1/8,\ U_{ac} = -3/4,\ \Omega = \epsilon,\ \mu_a = -\gamma/2 - 1/4,$
$\mu_c = \gamma/2 - 1/4,\ U_{ab} = U_{bc} = U_{bb} = \mu_b = 0.$
This model describes heteronuclear molecular BECs and is a generalisation of quantum optical models of Walls *et al* 1970, 1972. The BAE presented in Duncan *et al* 2007 are recovered after an appropriate substitution of variables.

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In this way our work above has provided an algebraic derivation of these models and their exact solutions.

THANKS!