Lower bounds on the state complexity of geometric Goppa codes *

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Abstract

We reinterpret the state space dimension equations for geometric Goppa codes. An easy consequence is that if deg $G \leq \frac{n-2}{2}$ or deg $G \geq \frac{n-2}{2} + 2g$ then the state complexity of $C_{\mathcal{L}}(D,G)$ is equal to the Wolf bound. For deg $G \in [\frac{n-1}{2}, \frac{n-3}{2} + 2g]$, we use Clifford's theorem to give a simple lower bound on the state complexity of $C_{\mathcal{L}}(D,G)$. We then derive two further lower bounds on the state space dimensions of $C_{\mathcal{L}}(D,G)$ in terms of the gonality sequence of $F/\mathbb{F}_q$. (The gonality sequence is known for many of the function fields of interest for defining geometric Goppa codes.) One of the gonality bounds uses previous results on the generalised weight hierarchy of $C_{\mathcal{L}}(D,G)$ and one follows in a straightforward way from first principles; often they are equal. For Hermitian codes both gonality bounds are equal to the DLP lower bound on state space dimensions. We conclude by using these results to calculate the DLP lower bound on state complexity for Hermitian codes.

Keywords. Geometric Goppa codes, Hermitian codes, state complexity, gonality sequence, dimension / length profiles, Clifford's theorem.

1 Introduction

Geometric Goppa codes have attracted much attention. They generalise Reed-Solomon codes and, although not maximum distance separable, can be longer than Reed-Solomon codes and have very good parameters for their lengths. The best known geometric Goppa codes, other than Reed-Solomon codes, are the Hermitian codes.

Let C be a linear code of length n. Many soft-decision decoding algorithms, such as the Viterbi algorithm, take place along a minimal trellis for C. The speed of such a decoding algorithm is determined by the structure of the trellis. The state complexity $\mathrm{s}(C)$ of C is the most-used trellis feature for measuring the complexity of trellis decoding algorithms for C. Therefore it is desirable that $\mathrm{s}(C)$ be small. A well-known upper bound on $\mathrm{s}(C)$ is the Wolf bound $\mathrm{W}(C) = \min\{\dim(C), n - \dim(C)\}$. It is well-known that $\mathrm{s}(C) = \mathrm{W}(C)$ if C is a Reed-Solomon code.

Recall that a trellis for C has its vertices placed at $n+1$ depths, here labelled $0, \ldots, n$. The number of vertices at each depth is a key determinant of the complexity of Viterbi-like algorithms. The code C has a minimal trellis with the least possible number of vertices at each depth. The set of vertices at each depth i of a minimal trellis forms a vector space, the dimension of which is called the state space dimension at depth i, denoted $\mathrm{s}_i(C)$. Then $\mathrm{s}(C) = \max\{\mathrm{s}_i(C) \ : \ 0 \leq i \leq n\}$. Also, $\mathrm{s}_i(C)$ can be characterised in terms of the dimensions of past- and future-punctured codes of C.

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We follow the notation of [10] for a geometric Goppa code, written $C_{\mathcal{L}}(D,G)$. Thus D, G are in the divisor group $\mathcal{D}_F$ of a function field $F/\mathbb{F}_q$, $D = \sum_{j=1}^{n} P_j$ where $P_1, \ldots, P_n$ are places of degree one and $k = \dim G - \dim(G-D)$. The genus of $F/\mathbb{F}_q$ is always denoted g. Hermitian codes are defined using $D, G \in \mathcal{D}_H$, where $H/\mathbb{F}_{q^2}$ is the Hermitian function field, $D = \sum_{j=1}^{q^3} Q_j$, $G = mQ_\infty$ and $Q_1, \ldots, Q_{q^3}, Q_\infty$ are the places of $H/\mathbb{F}_q$ of degree one. Section 2 is a table of frequently-used notation.

We determine lower bounds on $\mathrm{s}(C_{\mathcal{L}}(D,G))$. Our lower bounds imply the known bad behaviour of state complexity for Reed-Solomon codes, but allow $\mathrm{s}(C_{\mathcal{L}}(D,G)) < \mathrm{W}(C_{\mathcal{L}}(D,G))$ for certain geometric Goppa codes, including some Hermitian codes. The scope of this paper does not allow for calculating the actual state complexity of any of these codes, but elsewhere we show that our lower bounds can be tight, [3, 4].

We begin Section 3 by reinterpreting the characterisation of $\mathrm{s}_i(C_{\mathcal{L}}(D,G))$ in terms of the dimensions of past- and future-punctured codes. An easy consequence is that for $\deg G \leq \frac{n-2}{2}$ or $\deg G \geq \frac{n-2}{2} + 2g$, $\mathrm{s}(C_{\mathcal{L}}(D,G)) = \mathrm{W}(C_{\mathcal{L}}(D,G))$. In particular if $g = 0$ (so that $C_{\mathcal{L}}(D,G)$ is maximum distance separable) we recover the known result that $\mathrm{s}(C_{\mathcal{L}}(D,G))$ always reaches the Wolf bound. For $g > 0$ we are left to consider $\mathrm{s}(C_{\mathcal{L}}(D,G))$ for $\deg G \in I(n,g) = [\frac{n-1}{2}, \frac{n-3}{2} + 2g]$. We conclude Section 3 by using Clifford's theorem to give a simple lower bound on $\mathrm{s}(C_{\mathcal{L}}(D,G))$ when $\deg G \in I(n,g)$.

The dimension/length profile (DLP) of a code C can be used to bound the dimensions of past- and future-punctured codes of C and can thus be used to derive lower bounds on $\mathrm{s}(C)$, [6]. We write $\nabla_i(C)$ for the DLP bound for $\mathrm{s}_i(C)$ and $\nabla(C)$ for the DLP bound on $\mathrm{s}(C)$. The DLP of C is equivalent to the generalised weight hierarchy (GWH) of C.

Section 4 discusses the DLP of C and two gonality bounds. The DLP lower bound and the 'point of gain and fall' approach of [2, 5] suggest the notions of 'DLP points of gain and fall' (for use in Theorem 5.5).

In [13], the gonality sequence of a function field is introduced in order to determine a lower bound on the GWH of $C_{\mathcal{L}}(D,G)$. The first result of Section 4, Proposition 4.3, uses an improvement of the bound of [13], given in [8], to give a lower bound on $\nabla_i(C_{\mathcal{L}}(D,G))$ (and hence on $\mathrm{s}_i(C_{\mathcal{L}}(D,G))$) in terms of the gonality sequence of $F/\mathbb{F}_q$. Since it uses all this machinery, Proposition 4.3 could be considered to be quite deep. We then derive a second gonality sequence lower bound on $\mathrm{s}_i(C_{\mathcal{L}}(D,G))$ in a straightforward way from first principles. We show that the two gonality bounds are often equal.

We should note that the gonality sequence for many function fields of interest for defining geometric Goppa codes has been determined in [9]. In any case the gonality bounds can be used to easily recover the results of Section 3, except that the resulting 'Clifford bound' is slightly weaker than that of Section 3. A consequence of [9] is that the gonality sequence of $H/\mathbb{F}_{q^2}$ is equal to the pole number sequence of Q_∞. Thus we conclude Section 4 by expressing the gonality bounds for Hermitian codes in terms of the pole number sequence of Q_∞.

The equality of the gonality sequence of $H/\mathbb{F}_{q^2}$ and the pole number sequence of Q_∞ and results of [13] imply that, for Hermitian codes, the gonality bounds of Section 4 are equal to the DLP bound. Thus, we begin Section 5 by characterising $\nabla_i(C_{\mathcal{L}}(D,mQ_\infty))$ in terms of the pole number sequence of Q_∞ (Proposition 5.1). We use this characterisation of $\nabla_i(C_{\mathcal{L}}(D,mQ_\infty))$ and DLP points of gain and fall to determine $\nabla(C_{\mathcal{L}}(D,mQ_\infty))$ explicitly (Theorem 5.5, Corollary 5.6). This is always below the Wolf bound. In [4], we showed that $\mathrm{s}(C_{\mathcal{L}}(D,mQ_\infty)) = \nabla(C_{\mathcal{L}}(D,mQ_\infty))$ for over half of $m \in I(n,g)$, so that the gonality bounds of Section 4 can be tight in a non-trivial way.

The GWH of all Hermitian codes has been completely determined in [1], so that Proposition 5.1 could also be proved using the results of [1]. A summary of this paper and [4] appeared in [3].

2 Frequently Used-Notation

$F/\mathbb{F}_q$	Function field of one variable over $\mathbb{F}_q$
g	Genus of $F/\mathbb{F}_q$
$I(n,g)$	$[\frac{n-1}{2}, \frac{n-3}{2}+2g]$
$J(n,g)$	$[\frac{n-1}{2}, \frac{n-2}{2}+g]$
$\mathcal{D}_F$	The divisor group of $F/\mathbb{F}_q$
$\deg G,\ \dim G$	The degree and dimension of $G \in \mathcal{D}_F$
$P_1,\ldots,P_n$	Places of degree one in $F/\mathbb{F}_q$
D	$\sum_{j=1}^{n} P_j$
$C_{\mathcal{L}}(D,G)$	Geometric Goppa code over $\mathbb{F}_q$ with length n, dimension k and abundance a
$\mathrm{s}_i(C),\ \mathrm{s}(C)$	State space dimension at depth i and state complexity of C
$(d_r)_{r=1}^{k}$	Generalised weight hierarchy (GWH) of $[n,k]$ code C
$\nabla_i(C),\ \nabla(C)$	DLP bounds on state space dimensions and state complexity of C
$[b,c]$	$\{b, b+1, \ldots, c\}$
$\Gamma : \mathbb{N} \longrightarrow \mathbb{N}\cup\{0\}$	Gonality sequence of $F/\mathbb{F}_q$
$L_i^{(1)}(C_{\mathcal{L}}(D,G))$ $L^{(1)}(C_{\mathcal{L}}(D,G))$	Gonality bounds of Definition 4.2
$L_i^{(2)}(C_{\mathcal{L}}(D,G))$ $L^{(2)}(C_{\mathcal{L}}(D,G))$	Gonality bounds of Definition 4.6
$H/\mathbb{F}_{q^2}$	Hermitian function field
Q_∞	Place of degree one at infinity of $H/\mathbb{F}_{q^2}$
$Q_1,\ldots,Q_{q^3}$	Other places of degree one of $H/\mathbb{F}_{q^2}$
D	Also $\sum_{j=1}^{q^3} Q_j$
$C_{\mathcal{L}}(D,mQ_\infty)$	Hermitan code over $\mathbb{F}_{q^2}$ with length q^3, dimension k and abundance a
$\Pi : \mathbb{N} \longrightarrow \mathbb{N}\cup\{0\}$	Pole number sequence of Q_∞.

By a slight abuse of notation we allow the notation for $F/\mathbb{F}_q$ to carry over to $H/\mathbb{F}_{q^2}$, as we have for D above. So the genus of $H/\mathbb{F}_{q^2}$ is also denoted g, when considering $C_{\mathcal{L}}(D,mQ_\infty)$ we have $n=q^3$, etc.

3 Initial Results

Preliminaries. We begin this section with a more formal description of state complexity and geometric Goppa codes.

Let $0 \le i \le n$ and let C be a linear code of length n. The state space dimension at depth i of C is

$$\mathrm{s}_i(C) = \dim(\mathrm{Pu}_{i,-}(C)) + \dim(\mathrm{Pu}_{i,+}(C)) - \dim(C), \tag{1}$$

where

$$\mathrm{Pu}_{i,-}(C) = \{(c_1,\ldots,c_i) : (c_1,\ldots,c_n) \in C\}$$

and

$$\mathrm{Pu}_{i,+}(C) = \{(c_{i+1},\ldots,c_n) : (c_1,\ldots,c_n) \in C\}$$

are respectively the ith past- and future-punctured codes of C. The state complexity of C is then

$$\mathrm{s}(C) = \max\{\mathrm{s}_i(C) : 0 \le i \le n\}.$$

A well-known upper bound on $\mathrm{s}(C)$ is the Wolf bound, $\mathrm{W}(C) = \min\{\dim(C), n-\dim(C)\}$.

Our terminology and notation for geometric Goppa codes will for the most part follow [10]. Throughout, $F/\mathbb{F}_q$ is a function field (of one variable) over $\mathbb{F}_q$ with genus g, $\mathbb{P}_F$ is the set of places of $F/\mathbb{F}_q$ and $\mathcal{D}_F$ is the divisor group of $\mathbb{F}_q$. We assume throughout that $F/\mathbb{F}_q$ has a place of degree one. For $P \in \mathbb{P}_F$ and $x \in F/\mathbb{F}_q$ we have the valuation of x at P, $v_P(x)$. For $G \in \mathcal{D}_F$ we have the associated linear subspace of $F/\mathbb{F}_q$,

$$\mathcal{L}(G) = \{x \in F/\mathbb{F}_q : (x) \geq -G\} \cup \{0\}.$$

Let $P_1, \ldots, P_n$ be pairwise distinct places of $F/\mathbb{F}_q$ of degree one and $D = \sum_{j=1}^{n} P_j$. A geometric Goppa code is defined by means of D and a divisor G of $F/\mathbb{F}_q$ such that $\mathrm{supp}(D) \cap \mathrm{supp}(G) = \emptyset$. The geometric Goppa code associated with D and G is

$$C_{\mathcal{L}}(D, G) = \{x(P_1), \ldots, x(P_n) : x \in \mathcal{L}(G)\}.$$

We note that $C_{\mathcal{L}}(D, G)$ is a linear code over $\mathbb{F}_q$. The abundance of $C_{\mathcal{L}}(D, G)$ is $a = \dim(G - D)$ and if $a > 0$ then $C_{\mathcal{L}}(D, G)$ is called abundant, [8]. It is well-known that the dimension, k, of $C_{\mathcal{L}}(D, G)$ is given by $k = \dim G - a$ and that the minimum distance, d, of $C_{\mathcal{L}}(D, G)$ satisfies $d > n - \deg\, G$. If $\deg\, G < n$ (so that the bound on d has some meaning) then $a = 0$ (so that $k = \dim G$). We always have $\dim G \geq \deg\, G - g + 1$ and if $\deg\, G > 2g - 2$ then there is equality, by the Riemann-Roch theorem. It is usual to pay particular attention to the cases that $a = 0$ and/or $\deg\, G > 2g - 2$. In any case we are only interested in $0 \leq \deg\, G \leq n + 2g - 2$, since for $\deg\, G < 0$, $C_{\mathcal{L}}(D, G) = \{0\}$ and for $\deg\, G > n + 2g - 2$, $C_{\mathcal{L}}(D, G) = \mathbb{F}_q^n$ (and the trellises for these trivial codes are themselves trivial). *Throughout, the geometric Goppa code associated with D and G, $C_{\mathcal{L}}(D, G)$, has length n, dimension k and abundance a.*

First Consequences. We begin by reinterpreting (1) for geometric Goppa codes. For $0 \leq i \leq n$, we put $D_{i,-} = \sum_{j=1}^{i} P_j$ and $D_{i,+} = \sum_{j=i+1}^{n} P_j$, where an empty sum is taken to be the zero divisor. Then our first observation is that the past- and future-punctured codes of $C_{\mathcal{L}}(D, G)$ are again geometric Goppa codes and

$$\mathrm{Pu}_{i,-}(C_{\mathcal{L}}(D, G)) = C_{\mathcal{L}}(D_{i,-}, G) \quad \text{and} \quad \mathrm{Pu}_{i,+}(C_{\mathcal{L}}(D, G)) = C_{\mathcal{L}}(D_{i,+}, G). \tag{2}$$

Thus (1) becomes

$$\begin{aligned} \mathrm{s}_i(C_{\mathcal{L}}(D, G)) &= \dim(C_{\mathcal{L}}(D_{i,-}, G)) + \dim(C_{\mathcal{L}}(D_{i,+}, G)) - \dim(C_{\mathcal{L}}(D, G)) \\ &= \dim G + a - \dim(G - D_{i,-}) - \dim(G - D_{i,+}) \\ &= \dim(C_{\mathcal{L}}(D, G)) + 2a - \dim(G - D_{i,-}) - \dim(G - D_{i,+}). \end{aligned} \tag{3}$$

A simple consequence of (3) is

PROPOSITION 3.1 *If $0 \leq \deg\, G \leq \frac{n-2}{2}$, then $\mathrm{s}(C_{\mathcal{L}}(D, G)) = k$.*

PROOF. We always have $\mathrm{s}(C_{\mathcal{L}}(D, G)) \leq k$. Also, by hypothesis, $\deg\, G + 1 \leq n - \deg\, G - 1$, and for $\deg\, G + 1 \leq i \leq n - \deg\, G - 1$, (3) becomes $\mathrm{s}_i(C_{\mathcal{L}}(D, G)) \geq k$. □

Now $C_{\mathcal{L}}(D, G)^{\perp} = C_{\mathcal{L}}(D, H)$ for some divisor H with $\deg\, H = n - \deg\, G + 2g - 2$, [10, Corollary I.5.16, Proposition II.2.10]. Also $\mathrm{s}(C) = \mathrm{s}(C^{\perp})$, e.g. [6, Theorem 7]. Since $0 \leq \deg\, H \leq \frac{n-2}{2}$ if and only if $\frac{n-2}{2} + 2g \leq \deg\, G \leq n + 2g - 2$, we get

PROPOSITION 3.2 *If $\frac{n-2}{2} + 2g \leq \deg\, G \leq n + 2g - 2$ then $\mathrm{s}(C_{\mathcal{L}}(D, G)) = n - k$.*

It follows from Propositions 3.1 and 3.2 and the Wolf bound that if $0 \leq \deg\, G \leq \frac{n-2}{2}$ then $k \leq \frac{n}{2}$ and if $\frac{n-2}{2} + 2g \leq \deg\, G \leq n + 2g - 2$ then $k \geq \frac{n}{2}$. In fact we can show this directly from the conditions on $\deg\, G$. First, $0 \leq \deg\, G \leq \frac{n-2}{2}$ implies that $\dim(C_{\mathcal{L}}(D, G)) = \dim G \leq \frac{n}{2}$ by [10, Equation I(4.4)]. Next $\deg\, G \geq \frac{n-2}{2} + 2g$ implies that $\dim G \geq \frac{n}{2} + g$ by the definition of genus. Also $\deg\, G \leq n + 2g - 2$ implies that $\deg\, (G - D) \leq 2g - 2$, so by Clifford's theorem $\dim(G - D) \leq g$. Thus, for $\frac{n-2}{2} + 2g \leq \deg\, G \leq n + 2g - 2$, $\dim(C_{\mathcal{L}}(D, G)) \geq \frac{n}{2} + g - g = \frac{n}{2}$. Propositions 3.1 and 3.2 can be summarised as

Proposition 3.3 *For* $\deg G \in [0, \frac{n-2}{2}] \cup [\frac{n-2}{2} + 2g, n + 2g - 2]$, $\mathrm{s}(C_{\mathcal{L}}(D,G)) = \mathrm{W}(C_{\mathcal{L}}(D,G))$.

In particular if $g = 0$ (so that $C_{\mathcal{L}}(D,G)$ is MDS), we have the known result that $\mathrm{s}(C_{\mathcal{L}}(D,G))$ always reaches the Wolf bound.

The Clifford Bound. In view of Proposition 3.3 we need to determine $\mathrm{s}(C_{\mathcal{L}}(D,G))$ when $\deg G \in I(n,g)$. Here we give a simple numerical lower bound on $\mathrm{s}(C_{\mathcal{L}}(D,G))$ for $\deg G$ in this range. The proof of this lower bound is an application of Clifford's theorem so we refer to it as the Clifford bound.

Proposition 3.4 (Clifford Bound) *If* $\deg G \in I(n,g)$, *then*

$$\mathrm{s}(C_{\mathcal{L}}(D,G)) \geq k + 2a - \deg G + \left\lceil \frac{n-3}{2} \right\rceil.$$

Moreover if n *is odd,* $F/\mathbb{F}_q$ *is not hyperelliptic and* $\deg G \notin \{\frac{n-1}{2}, \frac{n+1}{2}, \frac{n-5}{2} + 2g, \frac{n-3}{2} + 2g\}$ *then*

$$\mathrm{s}(C_{\mathcal{L}}(D,G)) \geq k + 2a - \deg G + \frac{n-1}{2}.$$

Proof. We begin by recalling that Clifford's theorem states that for $A \in \mathcal{D}_F$ with $-1 \leq \deg A \leq 2g-1$,

$$\dim A \leq 1 + \frac{1}{2}\deg A.$$

(In fact Clifford's theorem is normally only stated for $0 \leq \deg A \leq 2g-2$ but is easily seen to hold when $\deg A = -1$ and $\deg A = 2g-1$.) The version of Clifford's theorem given in [7] also gives that the inequality is strict unless A is (i) a principle divisor or (ii) a canonical divisor or (iii) $F/\mathbb{F}_q$ is hyperelliptic (and A is an hyperelliptic divisor).

We first show that we can apply Clifford's theorem to $G - D_{i,-}$ and to $G - D_{i,+}$ whenever $\lfloor \frac{n-1}{2} \rfloor \leq i \leq \lfloor \frac{n+1}{2} \rfloor$. Now for $\lfloor \frac{n-1}{2} \rfloor \leq i \leq \lfloor \frac{n+1}{2} \rfloor$, we have $n - \deg G - 1 \leq i \leq \deg G + 1$ (since $\deg G \geq \lceil \frac{n-1}{2} \rceil$) so that $\deg (G - D_{i,-}) \geq -1$ and $\deg (G - D_{i,+}) \geq -1$. Also we have $\deg G - 2g + 1 \leq i \leq n - \deg G + 2g - 1$ (since $\deg G \leq \lfloor \frac{n-3}{2} \rfloor + 2g$) so that $\deg (G - D_{i,-}) \leq 2g-1$ and $\deg (G - D_{i,+}) \leq 2g-1$. For n even we can choose i, $\frac{n}{2} - 1 \leq i \leq \frac{n}{2} + 1$ such that both of $\deg G - i$ and $\deg G - n + i$ are odd (if $\deg G$ is even then choose i odd and if $\deg G$ is odd then choose i even). Then the inequality in Clifford's Theorem is strict for $A = G - D_{i,-}$ and for $A = G - D_{i,+}$ and (3) gives

$$\mathrm{s}_i(C_{\mathcal{L}}(D,G)) \geq k + 2a - \left(\frac{\deg G - i + 1}{2} + \frac{\deg G - n + i + 1}{2} \right) = k + 2a - \deg G + \frac{n-2}{2}.$$

For n odd we have that one of $\deg G - i$ and $\deg G - n + i$ is odd and one is even and for simplicity we can choose which one by taking either $i = \frac{n-1}{2}$ or $i = \frac{n+1}{2}$. We take $\deg G - i$ even. Thus, from (3),

$$\mathrm{s}_i(C_{\mathcal{L}}(D,G)) \geq k + 2a - \left(\frac{\deg G - i + 2}{2} + \frac{\deg G - n + i + 1}{2} \right) = k + 2a - \deg G + \frac{n-3}{2}.$$

Moreover if $\deg G \notin \{\frac{n-1}{2}, \frac{n+1}{2}, \frac{n-5}{2} + 2g, \frac{n-3}{2} + 2g\}$ then $\deg G - i \neq 0$ and $\deg G - i \neq 2g - 2$ so that $G - D_{i,-}$ is neither principle nor canonical. Thus if $F/\mathbb{F}_q$ is not hyperelliptic then

$$\mathrm{s}_i(C_{\mathcal{L}}(D,G)) \geq k + 2a - \left(\frac{\deg G - i}{2} + \frac{\deg G - n + i + 1}{2} \right) = k + 2a - \deg G + \frac{n-1}{2}.$$

□

In the case that $2g - 2 < \deg G < n$, the Clifford bound can be simplified to

Corollary 3.5 (Clifford Bound) *If* $\max\{\frac{n-1}{2}, 2g-1\} \le \deg G \le \min\{\frac{n-3}{2}+2g, n-1\}$ *then*

$$\mathrm{s}(C_{\mathcal{L}}(D,G)) \ge \left\lceil \frac{n-1}{2} \right\rceil - g.$$

Moreover if n *is odd,* $F/\mathbb{F}_q$ *is not hyperelliptic and* $\deg G \notin \{\frac{n-1}{2}, \frac{n+1}{2}, \frac{n-5}{2}+2g, \frac{n-3}{2}+2g\}$ *then*

$$\mathrm{s}(C_{\mathcal{L}}(D,G)) \ge \frac{n+1}{2} - g.$$

Proof. For $2g-2 < \deg G < n$, $\dim G = \deg G - g + 1$ and $\dim(G-D) = 0$. The result follows from Proposition 3.4. □

4 Gonality Bounds

In this section we determine two lower bounds on $\mathrm{s}_i(C_{\mathcal{L}}(D,G))$ for $0 \le i \le n$ in terms of the gonality sequence of $F/\mathbb{F}_q$. The gonality sequence was introduced in [13] to give a lower bound on the generalised weight hierarchy (GWH) of $C_{\mathcal{L}}(D,G)$. The GWH of a length n linear code C is equivalent to the dimension/length profile (DLP) of C, [6]. It is known that the DLP of C can be used to underbound $\mathrm{s}_i(C)$ for $0 \le i \le n$, [6] again. Therefore, our first gonality sequence lower bound on $\mathrm{s}_i(C_{\mathcal{L}}(D,G))$ is an application of the results of [8, 13] on the GWH of $C_{\mathcal{L}}(D,G)$. The second gonality sequence lower bound is derived from first principles in a straightforward way and is often equal to the first bound.

Gonality Sequence. As usual $\mathbb{N} = \{1,2,3,\ldots\}$ and for $b, c \in \mathbb{N} \cup \{0\}$ we put $[b,c] = \{b, b+1, \ldots, c\}$. The gonality sequence, $\Gamma : \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$, of $F/\mathbb{F}_q$ is defined in [13] by

$$\Gamma(r) = \min\{\deg A : A \in \mathcal{D}_F \text{ and } \dim A \ge r\}.$$

We will also need the translate Γ_b of Γ, given by $\Gamma_b(r) = \Gamma(r+b)$, for $b \in \mathbb{N} \cup \{0\}$. In [13], γ_r is used rather than $\Gamma(r)$. However, we use γ_i^{DLP} below in a different context. Also, the function notation is useful since we are particularly interested in sets of the form $\{r : \Gamma_b(r) \le R\} = \Gamma_b^{-1}[0,R]$. It is obvious from the definition of Γ that $\Gamma(\dim A) \le \deg A$.

Clearly Γ is non-decreasing, which implies that $|\Gamma^{-1}[0,R]| = \max(\Gamma^{-1}[0,R])$. In fact [13, Proposition 11] implies that Γ is increasing, since $F/\mathbb{F}_q$ contains a place of degree one. In particular Γ is injective and so Γ^{-1} can be regarded as the inverse of Γ on $\mathrm{Im}(\Gamma)$ in the natural way (by composing it with the map $2^{\mathbb{N}} \longrightarrow \mathbb{N}$ taking $\{r\}$ to r). Moreover, since Γ is increasing on $\mathbb{N}$, Γ^{-1} is increasing on $\mathrm{Im}(\Gamma)$. Using these facts we get

Lemma 4.1 *Since* $F/\mathbb{F}_q$ *contains a place of degree one,* $\dim A \le |\Gamma^{-1}[0, \deg A]|$ *for all* $A \in \mathcal{D}_F$.

Proof. We put $R_A = \max\{R \in \mathrm{Im}(\Gamma) : R \le \deg A\}$. Since Γ is increasing, we have

$$\Gamma^{-1}(R_A) = \max(\Gamma^{-1}[0, \deg A]) = |\Gamma^{-1}[0, \deg A]|.$$

Now, $\Gamma(\dim A) \le \deg A$ implies that $\Gamma(\dim A) \le R_A$. Thus, since $\Gamma^{-1} : \mathrm{Im}(\Gamma) \longrightarrow \mathbb{N}$ is increasing we have

$$\dim A = \Gamma^{-1}(\Gamma(\dim A)) \le \Gamma^{-1}(R_A) = |\Gamma^{-1}[0, \deg A]|,$$

as required. □

GWHs, DLPs and the First Gonality Bound. The dimension/length profile (DLP) and inverse DLP of a linear code were introduced in [6]. We take an $[n,k]$ code, C. If $J \subseteq \{1,\ldots,n\}$ with $|J| = i$ then we put

$$C_J = \{(c_1, \ldots, c_n) \in C : c_j = 0 \text{ for all } j \notin J\}$$

and

$$\mathrm{Pu}_J(C) = \{(c_{j_1}, \ldots, c_{j_i}) : (c_1, \ldots, c_n) \in C \text{ and } j_1, \ldots, j_i \in J\}.$$

The DLP of C is $(k_0(C), \ldots, k_n(C))$, where $k_i(C) = \max\{\dim(C_J) : |J| = i\}$ and the inverse DLP of C is $(\tilde{k}_0(C), \ldots, \tilde{k}_n(C))$, where $\tilde{k}_i(C) = \min\{\dim(\mathrm{Pu}_J(C)) : |J| = i\}$, [6]. (We note that the punctured code, $\mathrm{Pu}_J(C)$, is isomorphic to Forney's projected code, $P_J(C)$.) The DLP and inverse DLP are related by $\tilde{k}_{n-i}(C) = k - k_i(C)$, [6, Theorem 2]. Clearly $\dim(\mathrm{Pu}_{i,-}(C)) \geq \tilde{k}_i(C)$ and $\dim(\mathrm{Pu}_{i,+}(C)) \geq \tilde{k}_{n-i}(C)$. Thus we have from (1) that for $0 \leq i \leq n$,

$$\mathrm{s}_i(C) \geq \tilde{k}_i(C) + \tilde{k}_{n-i}(C) - k = \tilde{k}_i(C) - k_i(C). \tag{4}$$

The right-hand side of (4), which we denote by $\nabla_i(C)$, is the DLP bound on $\mathrm{s}_i(C)$, [6, Theorem 8]. The DLP bound on $\mathrm{s}(C)$ is then given by

$$\mathrm{s}(C) \geq \nabla(C) := \max\{\nabla_i(C) : 0 \leq i \leq n\}.$$

The generalised weight hierarchy (GWH), $1 \leq d_1 < \cdots < d_k \leq n$, of C was introduced in [12]. It can be defined in terms of the DLP of C as follows. First we note that for $1 \leq j \leq n$, $k_j(C) = k_{j-1}(C)+1$ for exactly k values of j (and otherwise $k_j(C) = k_{j-1}C$). Then, for $1 \leq r \leq k$, d_r can be defined as the rth value of j such that $k_j(C) = k_{j-1}(C) + 1$, see [6]. We note also, since $\tilde{k}_j(C) = k - k_{n-j}(C)$, [6, Theorem 2], that $\tilde{k}_j(C) = \tilde{k}_{j-1}(C) + 1$ if and only if $j = n - d_r + 1$ for some $1 \leq r \leq k$. Thus, led by (4) and the terminology of [2], we refer to a j for which $j = n - d_r + 1$ for some $1 \leq r \leq k$ as a *DLP point of gain* and a j for which $j = d_r$ for some $1 \leq r \leq k$ as *DLP point of fall*. (These concepts will be of use to us in Section 5.) We put

$$\gamma_i^{\mathrm{DLP}}(C) = |\{j : j \text{ is a DLP point of gain of } C \text{ and } j \leq i\}|$$

and

$$\delta_i^{\mathrm{DLP}}(C) = |\{j : j \text{ is a DLP point of fall of } C \text{ and } j \leq i\}|.$$

Since $\tilde{k}_0(C) = k_0(C) = 0$ we have that $\tilde{k}_i(C) = \gamma_i^{\mathrm{DLP}}(C)$ and $k_i(C) = \delta_i^{\mathrm{DLP}}(C)$. Then, from (4),

$$\begin{aligned} \nabla_i(C) &= \gamma_i^{\mathrm{DLP}}(C) - \delta_i^{\mathrm{DLP}}(C) = |\{r : n - d_r + 1 \leq i\}| - |\{r : d_r \leq i\}| \\ &= k - |\{r : d_r \leq n - i\}| - |\{r : d_r \leq i\}|. \end{aligned} \tag{5}$$

In [13, Theorem 12], it is shown that the GWH of $C_{\mathcal{L}}(D, G)$ is underbounded by $d_r \geq n - \deg G + \Gamma(r)$, for $1 \leq r \leq k$. This is improved on for abundant codes in [8, Corollary 2] to

$$d_r \geq n - \deg G + \Gamma_a(r), \quad \text{for } 1 \leq r \leq k.$$

Thus, for $R \geq 0$,

$$\{r : d_r \leq R\} \subseteq \{r : n - \deg G + \Gamma_a(r) \leq R \text{ and } 1 \leq r \leq k\} = \Gamma_a^{-1}[0, \deg G + R - n] \cap [1, k], \tag{6}$$

which together with (5) suggests the following definition.

DEFINITION 4.2 *For $0 \leq i \leq n$, we put*

$$L_i^{(1)}(C_{\mathcal{L}}(D, G)) = k - |\Gamma_a^{-1}[0, \deg G - i] \cap [1, k]| - |\Gamma_a^{-1}[0, \deg G + i - n] \cap [1, k]|.$$

Also we put

$$L^{(1)}(C_{\mathcal{L}}(D, G)) = \max\{L_i^{(1)}(C_{\mathcal{L}}(D, G)) : 0 \leq i \leq n\}.$$

Summarising

Proposition 4.3 *For $0 \leq i \leq n$,*

$$\mathrm{s}_i(C_{\mathcal{L}}(D,G)) \geq \nabla_i(C_{\mathcal{L}}(D,G)) \geq L_i^{(1)}(C_{\mathcal{L}}(D,G)).$$

Hence

$$\mathrm{s}(C_{\mathcal{L}}(D,G)) \geq \nabla(C_{\mathcal{L}}(D,G)) \geq L^{(1)}(C_{\mathcal{L}}(D,G)).$$

Proof. The result follows from (5) and two applications of (6); one with $R = n - i$ and the other with $R = i$. □

For many function fields, it is straightforward to calculate $L_i^{(1)}(C_{\mathcal{L}}(D,G))$ from [9, Corollary 2.4], provided that a and k are known. It is also possible to underbound $\nabla_i(C_{\mathcal{L}}(D,G))$ from Proposition 4.3 even if a is not known and/or only a lower bound on k is known, since $|\Gamma_a^{-1}[0,R]| \leq |\Gamma^{-1}[0,R]|$ and $|\Gamma_a^{-1}[0,R] \cap [1,k]| \leq |\Gamma_a^{-1}[0,R]|$. In Example 4.10 we calculate $L^{(1)}(C_{\mathcal{L}}(D,G))$ when $F/\mathbb{F}_q$ is hyperelliptic and in Section 5 we calculate $L^{(1)}(C_{\mathcal{L}}(D,G))$ when $C_{\mathcal{L}}(D,G)$ is an Hermitian code.

The Second Gonality Bound. We derive a result similar to Proposition 4.3 without reference to GWHs. In fact, since deg $(G - D_{i,-}) = \deg G - i$ and deg $(G - D_{i,+}) = \deg G - (n-i)$, we immediately get from (3) and Lemma 4.1

Proposition 4.4 *For $0 \leq i \leq n$,*

$$\mathrm{s}_i(C_{\mathcal{L}}(D,G)) \geq k + 2a - |\Gamma^{-1}[0, \deg\, G - i]| - |\Gamma^{-1}[0, \deg\, G + i - n]|.$$

In the case that $a = 0$, the lower bound in Proposition 4.4 is easily modified to agree with that in Proposition 4.3.

Proposition 4.5 *If $a = 0$ then for $0 \leq i \leq n$,*

$$\mathrm{s}_i(C_{\mathcal{L}}(D,G)) \geq k - |\Gamma^{-1}[0, \deg\, G - i] \cap [1,k]| - |\Gamma^{-1}[0, \deg\, G + i - n] \cap [1,k]|.$$

Proof. If $a = 0$ then $k = \dim G$ so that $\dim(G - D_{i,-}) \leq k$ and $\dim(G - D_{i,+}) \leq k$. Thus, from (3) and Lemma 4.1,

$$\mathrm{s}_i(C_{\mathcal{L}}(D,G)) \geq k - \min\{|\Gamma^{-1}[0, \deg\, G - i]|, k\} - \min\{|\Gamma^{-1}[0, \deg\, G - n + i]|, k\}.$$

□

We note that the proof of Proposition 4.5, like the proof of Proposition 4.4, essentially only uses (3) and Lemma 4.1 (and in particular is independent of Proposition 4.3). Thus it is natural to consider Propositions 4.4 and 4.5 together and so we make the following definition.

Definition 4.6 *For $0 \leq i \leq n$ we put*

$$L_i^{(2)}(C_{\mathcal{L}}(D,G)) = \begin{cases} k - |\Gamma^{-1}[0, \deg\, G - i] \cap [1,k]| - |\Gamma^{-1}[0, \deg\, G + i - n] \cap [1,k]| & \text{if } a = 0 \\ k + 2a - |\Gamma^{-1}[0, \deg\, G - i]| - |\Gamma^{-1}[0, \deg\, G + i - n]| & \text{if } a > 0. \end{cases}$$

Also we put

$$L^{(2)}(C_{\mathcal{L}}(D,G)) = \max\{L_i^{(2)}(C_{\mathcal{L}}(D,G)) : 0 \leq i \leq n\}.$$

Thus Propositions 4.4 and 4.5 yield $\mathrm{s}_i(C_{\mathcal{L}}(D,G)) \geq L_i^{(2)}(C_{\mathcal{L}}(D,G))$ for $0 \leq i \leq n$ and hence $\mathrm{s}(C_{\mathcal{L}}(D,G)) \geq L_i^{(2)}(C_{\mathcal{L}}(D,G))$.

Clearly if $a = 0$ then $L_i^{(1)}(C_{\mathcal{L}}(D,G)) = L_i^{(2)}(C_{\mathcal{L}}(D,G))$ for $0 \leq i \leq n$. Our next result shows that, even when $a > 0$, $L_i^{(1)}(C_{\mathcal{L}}(D,G))$ and $L_i^{(2)}(C_{\mathcal{L}}(D,G))$ usually agree.

PROPOSITION 4.7 *For* $\deg\, G - \Gamma(\dim G + 1) + 1 \leq i \leq \Gamma(\dim G + 1) - \deg\, G + n - 1$,

$$L_i^{(1)}(C_{\mathcal{L}}(D,G)) = L_i^{(2)}(C_{\mathcal{L}}(D,G)) = k + 2a - |\Gamma^{-1}[0, \deg\, G - i]| - |\Gamma^{-1}[0, \deg\, G + i - n]|.$$

In particular if $\Gamma(\dim G + 1) = \deg\, G + 1$ *then* $L_i^{(1)}(C_{\mathcal{L}}(D,G)) = L_i^{(2)}(C_{\mathcal{L}}(D,G))$ *for* $0 \leq i \leq n$.

PROOF. We first note that it suffices to show that, for i in the given range,

$$L_i^{(1)}(C_{\mathcal{L}}(D,G)) = k + 2a - |\Gamma^{-1}[0, \deg\, G - i]| - |\Gamma^{-1}[0, \deg\, G + i - n]|$$

(since for $a = 0$, $L_i^{(2)}(C_{\mathcal{L}}(D,G)) = L_i^{(1)}(C_{\mathcal{L}}(D,G))$).

Now $\dim G + 1 = k + a + 1$. Thus if $i \geq \deg\, G - \Gamma(\dim G + 1) + 1$ then $\Gamma_a(k+1) \geq \deg\, G - i + 1$ and if $i \leq \Gamma(\dim G + 1) - \deg\, G + n - 1$ then $\Gamma_a(k+1) \geq \deg\, G + i - n + 1$ and

$$L_i^{(1)}(C_{\mathcal{L}}(D,G)) = k - |\Gamma_a^{-1}[0, \deg\, G - i]| - |\Gamma_a^{-1}[0, \deg\, G + i - n]|.$$

Thus it remains to show that

$$|\Gamma_a^{-1}[0, \deg\, G - i]| + |\Gamma_a^{-1}[0, \deg\, G + i - n]| = |\Gamma^{-1}[0, \deg\, G - i]| + |\Gamma^{-1}[0, \deg\, G + i - n]| - 2a.$$

It is straightforward to see that if $|\Gamma^{-1}[0,R]| \geq a$ then $|\Gamma_a^{-1}[0,R]| = |\Gamma^{-1}[0,R]| - a$. Thus it suffices to show that $|\Gamma^{-1}[0, \deg\, G - i]| \geq a$ and $|\Gamma^{-1}[0, \deg\, G + i - n]| \geq a$. Using Lemma 4.1 we have, for $i \leq n$,

$$|\Gamma^{-1}[0, \deg\, G - i]| \geq |\Gamma^{-1}[0, \deg\, G - n]| \geq \dim(G - D) = a$$

and for $i \geq 0$,

$$|\Gamma^{-1}[0, \deg\, G + i - n]| \geq |\Gamma^{-1}[0, \deg\, G - n]| \geq \dim(G - D) = a,$$

which completes the proof. □

COROLLARY 4.8 *If* $\deg\, G > 2g - 2$ *then for* $0 \leq i \leq n$,

$$L_i^{(1)}(C_{\mathcal{L}}(D,G)) = L_i^{(2)}(C_{\mathcal{L}}(D,G)) = k + 2a - |\Gamma^{-1}[0, \deg\, G - i]| - |\Gamma^{-1}[0, \deg\, G + i - n]|.$$

PROOF. If $\deg\, G > 2g - 2$ then $\dim G + 1 = \deg\, G - g + 2 > g$, so that by [13, Proposition 11(b)], $\Gamma(\dim G + 1) = \dim G + g = \deg\, G + 1$. Thus the result follows from Proposition 4.7. □

We also note that if $\Gamma(\dim G + 1) < \deg\, G + 1$ then $\Gamma(\dim G) < \deg\, G$. Since, $\Gamma(\dim G) = \min\{\deg\, A : \dim A \geq \dim G\}$, this implies that there exists A with $\dim A \geq \dim G$ and $\deg\, A < \deg\, G$, so that $n - \deg\, A > n - \deg\, G$. Thus when constructing geometric Goppa codes, it is often best to choose G with $\Gamma(\dim G) = \deg\, G$.

We now give two examples. The first example shows that $L^{(2)}(C_{\mathcal{L}}(D,G))$ can be used to prove Proposition 3.3 and a slightly weaker version of the Clifford bound (Proposition 3.4 and Corollary 3.5). In the second example we show how $L^{(1)}(C_{\mathcal{L}}(D,G))$ and $L^{(2)}(C_{\mathcal{L}}(D,G))$ can be determined when the gonality sequence of $F/\mathbb{F}_q$ is known, by considering the case that $F/\mathbb{F}_q$ is hyperelliptic.

EXAMPLE 4.9 *(*cf. *Proposition 3.3 and Proposition 3.4):*

$$L^{(1)}(C_{\mathcal{L}}(D,G)) = L^{(2)}(C_{\mathcal{L}}(D,G)) = \begin{cases} k & \text{if } 0 \leq \deg\, G \leq \frac{n-2}{2} \\ n - k & \text{if } \frac{n-2}{2} + 2g - 2 \leq \deg\, G \leq n + 2g - 2. \end{cases}$$

Also, if $\deg\, G \in I(n,g)$ *then*

$$L^{(2)}(C_{\mathcal{L}}(D,G)) \geq k + 2a - \deg\, G + \left\lceil \frac{n-3}{3} \right\rceil.$$

PROOF. Firstly if deg $G \leq \frac{n-2}{2}$ then deg $G+1 \leq n-$ deg $G-1$ and taking deg $G+1 \leq i \leq n-$ deg $G-1$ we have

$$|\Gamma_a^{-1}[0, \deg G - i] \cap [1,k]| = |\Gamma_a^{-1}[0, \deg G - i]| = 0$$

and

$$|\Gamma_a^{-1}[0, \deg G + i - n] \cap [1,k]| = |\Gamma_a[0, \deg G + i - n]| = 0,$$

so that $L^{(1)}(C_{\mathcal{L}}(D,G)) = L^{(2)}(C_{\mathcal{L}}(D,G)) = k$.

Next we take deg $G \geq \frac{n-2}{2} + 2g$, so that $n -$ deg $G + 2g - 1 \leq$ deg $G - 2g + 1$. By Corollary 4.8 we have

$$L_i^{(1)}(C_{\mathcal{L}}(D,G)) = L_i^{(2)}(C_{\mathcal{L}}(D,G)) = k + 2a - |\Gamma^{-1}[0, \deg G - i]| - |\Gamma^{-1}[0, \deg G + i - n]|.$$

Now, by [13, Proposition 11], we have that for $R \geq 2g-1$, $|\Gamma^{-1}[0,R]| = R - g + 1$. Taking $n -$ deg $G + 2g - 1 \leq i \leq$ deg $G - 2g + 1$, we get deg $G - i \geq 2g - 1$ and deg $G + i - n \geq 2g - 1$, so that

$$L_i^{(2)}(C_{\mathcal{L}}(D,G)) = L_i^{(2)}(C_{\mathcal{L}}(D,G)) = k + 2a - 2\deg G + n + 2g - 2. \tag{7}$$

Since deg $G > 2g-2$, the Riemann-Roch theorem gives that $\dim G =$ deg $G - g + 1$, so that $k =$ deg $G - g + 1 - a$ and the right-hand side of (7) is $n - k$.

Finally we take deg $G \in I(n,g)$. The second inequality (which is all we use) implies that deg $G - 2g + 1 < n -$ deg $G + 2g - 1$. By [13, Proposition 11], we have that for $R \leq 2g-1$, $|\Gamma^{-1}[0,R]| \leq \lfloor \frac{R}{2} \rfloor + 1$. Taking deg $G - 2g + 1 \leq i \leq n -$ deg $G + 2g - 1$, we have deg $G - i \leq 2g - 1$ and deg $G + i - n \leq 2g - 1$ so that

$$L_i^{(2)}(C_{\mathcal{L}}(D,G)) \geq k + 2a - \left\lfloor \frac{\deg G - i}{2} \right\rfloor - \left\lfloor \frac{\deg G + i - n}{2} \right\rfloor - 2.$$

Now, for n even we can choose i, deg $G - 2g + 1 \leq i \leq n -$ deg $G + 2g - 1$ such that deg $G - i$ and deg $G + i - n$ are both odd, and for n odd, one of deg $G - i$ and deg $G + i - n$ is odd and one is even. Thus

$$L^{(2)}(C_{\mathcal{L}}(D,G)) \geq k + 2a - \deg G + \left\lceil \frac{n-3}{2} \right\rceil.$$

□

In Example 4.9, $L^{(1)}(C_{\mathcal{L}}(D,G))$ and $L^{(2)}(C_{\mathcal{L}}(D,G))$ are determined whenever deg $G \leq \frac{n-2}{2}$ or deg $G \geq \frac{n-2}{2} + 2g$. Thus we are left to determine them for deg $G \in I(n,g)$. In Example 4.10, we do this when $F/\mathbb{F}_q$ is hyperelliptic and $\Gamma(\dim G + 1) =$ deg $G + 1$.

EXAMPLE 4.10 *If $F/\mathbb{F}_q$ is hyperelliptic,* deg $G \leq \frac{n-3}{2} + 2g$ *and* $\Gamma(\dim G + 1) =$ deg $G + 1$ *then*

$$L^{(1)}(C_{\mathcal{L}}(D,G)) = L^{(2)}(C_{\mathcal{L}}(D,G)) = k + 2a - \deg G + \left\lceil \frac{n-3}{2} \right\rceil.$$

PROOF. We work from Proposition 4.7. In particular $L^{(1)}(C_{\mathcal{L}}(D,G)) = L^{(2)}(C_{\mathcal{L}}(D,G))$. We note that deg $G \leq \frac{n-3}{2} + 2g$ implies deg $G - 2g + 1 < n -$ deg $G + 2g - 1$. The gonality sequence of $F/\mathbb{F}_q$ is given by

$$\Gamma(r) = \begin{cases} 2r - 2 & \text{for } 1 \leq r \leq g \\ r + g - 1 & \text{for } r \geq g + 1, \end{cases}$$

[8, 11]. Thus

$$|\Gamma^{-1}[0,R]| = \begin{cases} \lfloor \frac{R}{2} \rfloor + 1 & \text{if } 0 \leq R \leq 2g - 1 \\ R - g + 1 & \text{if } R \geq 2g - 1, \end{cases}$$

and $L_i^{(2)}(C_{\mathcal{L}}(D,G))$ takes the following values:

$$\begin{cases} k+2a-\deg\,G+i+g-\lfloor\frac{\deg\,G+i-n}{2}\rfloor-2 & \text{for } 0\le i\le \deg\,G-2g-1\\ k+2a-\lfloor\frac{\deg\,G-i}{2}\rfloor-\lfloor\frac{\deg\,G+i-n}{2}\rfloor-2 & \text{for } \deg\,G+2g-1\le i\le n-\deg\,G+2g-1\\ k+2a-\lfloor\frac{\deg\,G-i}{2}\rfloor-\deg\,G-i+n+g-2 & \text{for } n-\deg\,G+2g-1\le i\le n.\end{cases}$$

It is straightforward to see that $L_i^{(2)}(C_{\mathcal{L}}(D,G))$ is maximised over $0\le i\le \deg\,G-2g-1$ at $i=\deg\,G-2g-1$ and is maximised over $n-\deg\,G+2g-1\le i\le n$ at $i=n-\deg\,G+2g-1$. Thus $L_i^{(2)}(C_{\mathcal{L}}(D,G))$ is maximised over $0\le i\le n$, for some $\deg\,G+2g-1\le i\le n-\deg\,G+2g-1$. Now, for even n, $L_i^{(2)}(C_{\mathcal{L}}(D,G))$ is maximised over $\deg\,G-2g-1\le i\le n-\deg\,G+2g-1$ by taking i such that both $\deg\,G-i$ and $\deg\,G+i-n$ are odd (noting that there exist both even and odd i in this range). For such i,

$$L_i^{(2)}(C_{\mathcal{L}}(D,G))=k+2a-\deg\,G+\frac{n-2}{2}.$$

For odd n, one of $\deg\,G-i$ and $\deg\,G+i-n$ will be odd and one even, so that for $\deg\,G-2g-1\le i\le n-\deg\,G+2g-1$,

$$L_i^{(2)}(C_{\mathcal{L}}(D,G))=k+2a-\deg\,G+\frac{n-3}{2}.$$

Thus

$$L^{(2)}(C_{\mathcal{L}}(D,G))=k+2a-\deg\,G+\left\lceil\frac{n-3}{2}\right\rceil,$$

and the proof is complete. □

We note that, when $F/\mathbb{F}_q$ is hyperelliptic and $\Gamma(\dim G+1)=\deg\,G+1$, the Clifford bound (Proposition 3.4) and both gonality bounds agree.

A pole number sequence and the gonality bounds for Hermitian codes. The Hermitian function field, $H/\mathbb{F}_{q^2}$ has genus $g=\binom{q}{2}$ and q^3+1 places of degree one. One of the places of degree one is the place at infinity, denoted Q_∞, and the other places of degree one we denote $Q_1,\ldots,Q_{q^3}$. The Hermitian codes are of the form $C_{\mathcal{L}}(D,mQ_\infty)$, where $D=\sum_{j=1}^{q^3}Q_j$. Thus Hermitian codes are defined over $\mathbb{F}_{q^2}$ and have $n=q^3$, which is much longer than Reed-Solomon codes over $\mathbb{F}_{q^2}$. As is usual, we write C_m for $C_{\mathcal{L}}(D,mQ_\infty)$. When m is understood we write k for $\dim(C_m)$.

We conclude this section by showing that, for Hermitian codes, the gonality bounds can be expressed in terms of the pole number sequence of Q_∞. We shall see in Section 5 that the DLP bound for Hermitian codes often has an identical expression.

From [10, Proposition VI.4.1], t is a pole number of Q_∞ if and only if $t=iq+j(q+1)$ for some $i\ge 0$ and $0\le j\le q-1$. This quickly translates to

Lemma 4.11 *The set of pole numbers of Q_∞ is $\{iq+j: 0\le i\le q-1, 0\le j\le i\}\cup\{i: i\ge 2g+q\}$.*

Proof. First we note that $iq+j=(i-j)q+j(q+1)$, so that if $0\le j\le i\le q-1$ then $iq+j$ is a pole number. Similarly for $t\ge 2g+q=q^2$, we can write $t=i'q+j=(i'-j)q+j(q+1)$, where $i'\ge q$ and $0\le j\le q-1\le i'$, so that t is a pole number.

For the reverse inclusion we take $t=i'q+j(q+1)$, with $i'\ge 0$ and $0\le j\le q-1$. For $0\le i'\le q-1-j$ we put $i=(i'+j)$ and have $t=iq+j$ where $0\le i\le q-1$ and $0\le j\le i$. For $i'\ge q-j$ we have $t=(i'+j)q+j\ge q^2=2g+q$. □

We write $\Pi:\mathbb{N}\longrightarrow\mathbb{N}\cup\{0\}$ for the pole number sequence of Q_∞. We note that $\dim G=|\Pi^{-1}[0,m]|$. Writing Γ for the gonality sequence of $H/\mathbb{F}_{q^2}$, it follows easily from Lemma 4.11 that for $r>g$,

$\Pi(r) = r + g - 1 = \Gamma(r)$, *cf.* [13, Remark 17(c) and Proposition 11]. In fact, it follows from [9, Corollary 2.4] that $\Pi = \Gamma$ and hence $|\Gamma^{-1}[0, R]| = |\Pi^{-1}[0, R]|$ for all $R \geq 0$. Thus Definitions 4.2 and 4.6 can be rephrased using Π. In fact

PROPOSITION 4.12 *For $0 \leq i \leq n$,*

$$L_i^{(1)}(C_m) = L_i^{(2)}(C_m) = k + 2a - |\Pi^{-1}[0, m - i]| - |\Pi^{-1}[0, m + i - n]|.$$

PROOF. First we take $m < n$, so that $a = 0$ and

$$L_i^{(1)}(C_m) = L_i^{(2)}(C_m) = k - \min\{|\Pi^{-1}[0, m - i]|, k\} - \min\{|\Pi^{-1}[0, m + i - n]|, k\}.$$

Also, for $0 \leq i \leq n$, $k = |\Pi^{-1}[0, m]| \geq |\Pi^{-1}[0, m - i]|$ and similarly $k \geq |\Pi^{-1}[0, m + i - n]|$.

The result for $m \geq n$ follows from Corollary 4.8. □

5 The DLP bound for Hermitian codes

Proposition 4.3 and Example 4.9 imply that $\nabla(C_m) = \mathrm{W}(C_m)$ for $m \in [0, \frac{n-2}{2}] \cup [\frac{n-2}{2} + 2g - 2, n + 2g - 2]$. In this section we calculate $\nabla(C_m)$ for $m \in I(n, g)$.

A first step is to show that for $m \in I(n, g)$, $\nabla_i(C_m)$ is equal the expression for the gonality bounds on $\mathrm{s}_i(C_m)$ calculated in Proposition 4.12. The proof of Proposition 5.1 uses the results of [13] on the GWH of Hermitian codes and the fact that $\Pi = \Gamma$, which follows from [9, Corollary 2.4], as noted in the last section.

PROPOSITION 5.1 *If $m \in [2g + q^2 - 2, n - 1]$ and $n - m$ is a pole number then for $0 \leq i \leq n$,*

$$\nabla_i(C_m) = k - |\Pi^{-1}[0, m - i]| - |\Pi^{-1}[0, m + i - n]|. \tag{8}$$

In particular (8) holds for $m \in I(n, g)$.

PROOF. First we recall that [13, Theorem 12] states that $d_r \geq n - m + \Gamma(r)$ for $1 \leq r \leq k$. Next [13, Theorem 21] states that, if $2g + q^2 - 2 \leq m < n$ and $n - m$ is a pole number then $d_r \leq n - m + \Pi(r)$ for $1 \leq r \leq g$. Also [13, Corollary 6] and $k = m - g + 1$ (since $m \geq 2g + q^2 - 2 \geq 2g - 2$) imply that $d_r = n - m + g - 1 + r = n - m + \Pi(r)$ for $g + 1 \leq r \leq k$. Thus, since $\Pi = \Gamma$, we have that

$$d_r = n - m + \Pi(r) \quad \text{for } 1 \leq r \leq k.$$

Thus (5) becomes

$$\nabla_i(C_m) = k - |\Pi^{-1}[0, m - i]| - |\Pi^{-1}[0, m + i - n]|.$$

It remains to show that if $m \in I(n, g)$ then $2g + q^2 - 2 \leq m < n$ and $n - m$ is a pole number. Noting that $g \geq 1$, it suffices to show that $2g + q^2 - 2 \leq m \leq n - 2g$ (since then $n - m \geq 2g$ is a pole number). First, $\frac{n-1}{2} \geq 2q^2 - q - 2$ if and only if $q^3 \geq 4q^2 - 2q - 3$. This is clearly true for $q \geq 4$ (since then $q^3 \geq 4q^2$) and holds with equality for $q = 3$. For $q = 2$ we have that $m \geq \frac{n-1}{2}$ implies that $m \geq \frac{n}{2} = 4$ and $2q^2 - q - 2 = 4$. Thus $m \geq 2g + q^2 - 2$ for $q \geq 2$. Next, $\frac{n-2}{2} + 2g \leq n - 2g$ if and only if $q^3 \geq 4q^2 - 4q - 2$. Again this clearly holds for $q \geq 4$ and can be checked directly for $q = 2$ and $q = 3$. Thus $m \leq n - 2g$ for $q \geq 2$. □

COROLLARY 5.2 *For $0 \leq m \leq n + 2g - 2$ and $0 \leq i \leq n$,*

$$\nabla_i(C_m) = L_i^{(1)}(C_m) = L_i^{(2)}(C_m).$$

In particular

$$\nabla(C_m) = L^{(1)}(C_m) = L^{(2)}(C_m).$$

Proof. This follows from Example 4.9 and Propositions 4.12 and 5.1. □

For the rest of this section we only refer to the DLP bound. We note here that Corollary 5.2 means that all results and calculations apply equally to the gonality bounds.

Examples 5.3 *1. For $q = 2$ we have $n = 8$ and $g = 1$, so that for $0 \leq m \leq 3$, $\nabla(C_m) = k$ and for $5 \leq m \leq 8$, $\nabla(C_m) = n - k$. Thus it remains to calculate* $\mathrm{s}(C_m)$ *for $m = 4$. From Lemma 4.11 we have $\Pi[1,4] = \{0,2,3,4\}$. Thus* $\dim(C_4) = |\Pi^{-1}[0,4]| = 4$ *and from Proposition 5.1 we have*

i	0	1	2	3	4	5	6	7	8
$\lvert\Pi^{-1}[0,4-i]\rvert$	4	3	2	1	1	0	0	0	0
$\lvert\Pi^{-1}[0,i-4]\rvert$	0	0	0	0	1	1	2	3	4
$\nabla_i(C_4)$	0	1	2	3	2	3	2	1	0

Thus $\nabla(C_4) = 3$ and $\nabla_i(C_4) = \nabla(C_4)$ for $i \in \{3,5\}$.

2. For $q = 3$ we have $n = 27$ and $g = 3$, so that for $0 \leq m \leq 12$, $\nabla(C_m) = k$ and for $19 \leq m \leq 31$, $\nabla(C_m) = n - k$. We note, by [10, Proposition VII.4.2], that $C_m^\perp = C_{31-m}$. Thus $C_{16}^\perp = C_{15}$, $C_{17}^\perp = C_{14}$ and $C_{18}^\perp = C_{13}$. Since $\nabla(C) = \nabla(C^\perp)$ by [6, Theorem 3], it remains to calculate $\nabla(C_m)$ for $m \in \{13,14,15\}$. Since $\nabla_i(C) = \nabla_{n-i}(C)$ (this is well-known or follows for Hermitian codes from Proposition 5.1), it suffices to find $\max\{\nabla_i(C_m) : 0 \leq i \leq 13\}$.

We have from Lemma 4.11 that $\Pi[1,13] = \{0,3,4,6,7,8,9,10,11,12,13,14,15\}$. Thus for $m = 13$ we have $k = |\Pi^{-1}[0,13]| = 11$ and from Proposition 5.1 we get,

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lvert\Pi^{-1}[0,13-i]\rvert$	11	10	9	8	7	6	5	4	3	3	2	1	1	1
$\lvert\Pi^{-1}[0,i-14]\rvert$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\nabla_i(C_{13})$	0	1	2	3	4	5	6	7	8	8	9	10	10	10

Thus, for $m = 13$ and $m = 18$, $\nabla(C_m) = 10$ and $\nabla_i(C_m) = \nabla(C_m)$ for $i \in [11,16]$.

For $m = 14$ we have $k = 12$ and

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lvert\Pi^{-1}[0,14-i]\rvert$	12	11	10	9	8	7	6	5	4	3	3	2	1	1
$\lvert\Pi^{-1}[0,i-13]\rvert$	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$\nabla_i(C_{13})$	0	1	2	3	4	5	6	7	8	9	9	10	11	10

.

Thus, for for $m = 14$ and $m = 17$, $\nabla(C_m) = 11$ and $\nabla_i(C_m) = \nabla(C_m)$ for $i \in \{12,15\}$.

Finally for $m = 15$ we have $k = 13$ and

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lvert\Pi^{-1}[0,15-i]\rvert$	13	12	11	10	9	8	7	6	5	4	3	3	2	1
$\lvert\Pi^{-1}[0,i-12]\rvert$	0	0	0	0	0	0	0	0	0	0	0	0	1	1
$\nabla_i(C_{13})$	0	1	2	3	4	5	6	7	8	9	10	10	10	11

.

Thus, for $m = 15$ and $m = 16$, $\nabla(C_m) = 11$ and $\nabla_i(C_m) = \nabla(C_m)$ for $i \in [13,14]$.

To calculate $\nabla(C_m)$ for $m \in [\frac{n-1}{2}, \frac{n-3}{2} + 2g - 2]$ from Proposition 5.1 we need to determine an i for which $\nabla_i(C_m)$ is maximised. To do this we use the notions of DLP points of gain and DLP points of fall introduced in Section 4. We put $\mathrm{DLP_{gain}}(m)$ and $\mathrm{DLP_{fall}}(m)$ respectively equal to the set of DLP points of gain of C_m and set of DLP points of fall of C_m.

Lemma 5.4 *For $m \in [\frac{n-1}{2}, \frac{n-3}{2} + 2g - 2]$,*

$$\mathrm{DLP_{gain}}(m) = [1, m - 2g - q + 1] \cup \{m - 2g - q + 2 + iq + j : 0 \leq i \leq q - 1, i \leq j \leq q - 1\}$$

and

$$\mathrm{DLP}_{\mathrm{fall}}(m) = \{n-m+2g+q-1-iq-j : 0 \le i \le q-1, i \le j \le q-1\} \cup [n-m+2g+q, n].$$

PROOF. As in the proof of Proposition 5.1, if $m \in I(n,g)$ then $d_r = n - m + \Pi(r)$ for $1 \le r \le k$. Thus, for $m \in I(n,g)$, $j \in \mathrm{DLP}_{\mathrm{gain}}(m)$ if and only if $j = m - \Pi(r) + 1$ for some $1 \le r \le k$ (if and only if $|\Pi^{-1}[0, m-j+1]| = |\Pi^{-1}[0, m-j]| + 1$) and $j \in \mathrm{DLP}_{\mathrm{fall}}(m)$ if and only if $j = n - m + \Pi(r)$ for some $1 \le r \le k$ (if and only if $|\Pi^{-1}[0, n-m+j]| = |\Pi^{-1}[0, n-m+j-1]| + 1$). Thus

$$\mathrm{DLP}_{\mathrm{gain}}(m) = \{m - \Pi(r) + 1 : 1 \le r \le k\} \quad \text{and} \quad \mathrm{DLP}_{\mathrm{fall}}(m) = n - \mathrm{DLP}_{\mathrm{gain}}(m) + 1.$$

Now, for $m < n$ we have $k = |\Pi^{-1}[0,m]|$ and so for $2g + q - 1 \le m < n$ we have from Lemma 4.11,

$$\Pi[1,k] = \{iq + j : 0 \le i \le q-1, 0 \le j \le i\} \cup [2g+q, m].$$

Thus, since $2g + q - 1 \le m < n$,

$$\begin{aligned}
\mathrm{DLP}_{\mathrm{gain}}(m) &= [1, m-2g-q+1] \cup \{m+1-i'q-j' : 0 \le i' \le q-1, 0 \le j' \le i'\} \\
&= [1, m-2g-q+1] \cup \{m-2g-q+2+iq+j : 0 \le i \le q-1, i \le j \le q-1\},
\end{aligned}$$

where the second equality follows with $i = q-1-i'$ and $j = q-1-j'$. Also then,

$$\mathrm{DLP}_{\mathrm{fall}}(m) = \{n-m+2g+q-1-iq-j : 0 \le i \le q-1, i \le j \le q-1\} \cup [n-m+2g+q, n].$$

□

To begin with we restrict our attention to $m \in J(n,g) = [\frac{n-1}{2}, \frac{n-2}{2} + g]$. The DLP bound for $m \in I(n,g)$ will follow since $\nabla_i(C) = \nabla_i(C^\perp)$ by [6, Theorem 3] and $C_m^\perp = C_{n+2g-2-m}$ by [10, Proposition VII.4.2] (as already noted in Example 5.3.2).

We recall from Section 4 that $\gamma_i^{\mathrm{DLP}}(C_m) = \mathrm{DLP}_{\mathrm{gain}}(m) \cap [1,i]$, $\delta_i^{\mathrm{DLP}}(C_m) = \mathrm{DLP}_{\mathrm{fall}}(m) \cap [1,i]$ and

$$\nabla_i(C_m) = \gamma_i^{\mathrm{DLP}}(C_m) - \delta_i^{\mathrm{DLP}}(C_m).$$

THEOREM 5.5 *For $m \in J(n,g)$ write $n - 2m + 4g + q - 2 = uq + v$, where $0 \le v \le q-1$. Then $\nabla(C_m)$ is attained at $m - 2g + 1 + \lfloor \frac{u}{2} \rfloor q$ and equals*

$$k - \binom{q - \lfloor \frac{u}{2} \rfloor}{2} - \binom{q - \lceil \frac{u}{2} \rceil}{2} - \min\left\{q - \left\lceil \frac{u}{2} \right\rceil, q - v\right\}.$$

PROOF. If $h \in \mathrm{DLP}_{\mathrm{gain}}(m)$ then $\mathrm{s}_h(C_m) \ge \mathrm{s}_{h-1}(C_m)$. Thus $\nabla(C_m)$ is attained (not necessarily uniquely) at an h for which $h \in \mathrm{DLP}_{\mathrm{gain}}(m)$ and $h+1 \notin \mathrm{DLP}_{\mathrm{gain}}(m)$. Therefore Lemma (5.4) implies that $\nabla(C_m)$ is attained at

$$h_i = m - 2g - q + 2 + iq + q - 1 = m - 2g + 1 + iq \quad \text{for some } 0 \le i \le q-1.$$

Also

$$\gamma_{h_i}^{\mathrm{DLP}}(C_m) = m - 2g - q + 1 + \sum_{j=0}^{i}(q-j) = m - g + 1 - \binom{q-i}{2}.$$

We also write $h_i = n - m + 2g + q - 1 - i'q - j'$ for some integers i' and $0 \le j' \le q-1$. Then, $(i+i')q + j' = n - 2m + 4g + q - 2$, so that with u and v as in the statement of the theorem, $i' = u - i$ and $j' = v$. Now for $m \in J(n,g)$,

$$q^2 = 2g + q \le n - 2m + 2g + q - 2 \le 4g + q - 1 = (2q-2)q + (q-1). \tag{9}$$

In particular, $u \geq q$, so that $i' > 0$. Now Lemma 5.4 implies that for $i' \geq q$, $\delta_{h_i}^{\text{DLP}} = 0$ and for $0 \leq i' \leq q-1$,

$$\delta_{h_i}^{\text{DLP}}(C_m) = \sum_{j=1}^{q-1-i'} j + \min\{q-i', q-j'\} = \binom{q-i'}{2} + \min\{q-i', q-j'\}.$$

Thus

$$\delta_{h_i}^{\text{DLP}} = \begin{cases} 0 & \text{if } 0 \leq i \leq u-q \\ \binom{q-u+i}{2} + \min\{q-u+i, q-v\} & \text{if } u-q+1 \leq i \leq q-1. \end{cases}$$

Hence with

$$\mu(i) = \begin{cases} \binom{q-i}{2} & \text{if } 0 \leq i \leq u-q \\ \binom{q-i}{2} + \binom{q-u+i}{2} + \min\{q-u+i, q-v\} & \text{if } u-q+1 \leq i \leq q-1, \end{cases}$$

we have

$$\nabla_{h_i}(C_m) = m - g + 1 - \mu(i).$$

Therefore to find an i, $0 \leq i \leq q-1$, that maximises $\nabla_{h_i}(C_m)$ (and hence an h that maximises $\nabla_h(C_m)$) it suffices to find an i, $0 \leq i \leq q-1$, that minimises $\mu(i)$.

Now

$$\mu(i) - \mu(i-1) = \begin{cases} i-q & \text{for } 1 \leq i \leq u-q \\ 2i-u & \text{for } u-q+1 \leq i \leq u-v \\ 2i-u-1 & \text{for } u-v+1 \leq i \leq q-1. \end{cases}$$

Thus, for $1 \leq i \leq u-q$, $\mu(i) < \mu(i-1)$ and $\mu(i)$ is minimised over $0 \leq i \leq u-q$ at $i = u-q$. Also for $u-q+1 \leq i \leq q-1$, $2i-u-1 \leq \mu(i) - \mu(i-1) \leq 2i-u$, so that, for $u-q+1 \leq i \leq \lfloor \frac{u}{2} \rfloor$, $\mu(i) - \mu(i-1) \leq 0$ and for $u+1 \leq \lfloor \frac{u}{2} \rfloor \leq i \leq q-1$, $\mu(i) - \mu(i-1) \geq 0$. Thus, provided $u-q+1 \leq \lfloor \frac{u}{2} \rfloor \leq q-1$, $\mu(i)$ is minimised over $u-q \leq i \leq q-1$ at $i = \lfloor \frac{u}{2} \rfloor$. Now from (9), we have $u \leq 2q-2$ so that $2u-2q+2 \leq u$ and hence $\lfloor \frac{u}{2} \rfloor \geq u-q+1$. Also $u \leq 2q-2$ implies that $\lfloor \frac{u}{2} \rfloor \leq q-1$. Thus $\mu(i)$ is minimised over $0 \leq i \leq q-1$ at $i = \lfloor \frac{u}{2} \rfloor$ and is equal to

$$\binom{q-\lfloor \frac{u}{2} \rfloor}{2} + \binom{q-\lceil \frac{u}{2} \rceil}{2} + \min\left\{q - \left\lceil \frac{u}{2} \right\rceil, q-v\right\}.$$

The theorem follows from the definitions of h_i and $\mu(i)$ and the fact that $m \geq \frac{n-1}{2} > 2g-2$ so that $k = m-g+1$ by the Riemann-Roch Theorem. □

Corollary 5.6 *For* $m \in [\frac{n-1}{2}+g, \frac{n-3}{2}+2g]$, *write* $2m-n+q+2 = uq+v$, *where* $0 \leq v \leq q-1$. *Then* $\nabla(C_m)$ *is attained at* $n-m-1+\lfloor \frac{u}{2} \rfloor q$ *and equals*

$$n - k - \binom{q-\lfloor \frac{u}{2} \rfloor}{2} - \binom{q-\lceil \frac{u}{2} \rceil}{2} - \min\left\{q - \left\lceil \frac{u}{2} \right\rceil, q-v\right\}.$$

Proof. We put $m^\perp = n+2g-2-m$, so that $C_m^\perp = C_{m^\perp}$. For $m \in [\frac{n-1}{2}+g, \frac{n-3}{2}+2g]$ we have $\frac{n-1}{2} \leq m^\perp \leq \frac{n-3}{2}+g$. We write

$$2m-n+q+2 = n-2m^\perp+4g+q-2 = uq+v.$$

Then by Theorem 5.5, and the fact that $\nabla_i(C^\perp) = \nabla_i(C)$, $\nabla(C_m)$ is attained at $m^\perp+2g+1+\lfloor \frac{u}{2} \rfloor q = n-m-1+\lfloor \frac{u}{2} \rfloor q$ and is equal to

$$\dim(C_m^\perp) - \binom{q-\lfloor \frac{u}{2} \rfloor}{2} - \binom{q-\lceil \frac{u}{2} \rceil}{2} - \min\left\{q - \left\lceil \frac{u}{2} \right\rceil, q-v\right\}.$$

□

EXAMPLES 5.7 *1. For $q = 2$ (so that $n = 8$ and $g = 1$), Theorem 5.5 applies to $m = 4$. For $m = 4$, $n - 2m + 4g + q - 2 = 4 = 2 \cdot 2$, so that $u = 2$ and $v = 0$. Thus from Theorem 5.5, $\nabla_5(C_4) = \nabla(C_4)$ and $\nabla(C_4) = 3$, both of which agree with Example 5.3.1.*

2. For $q = 3$ (so that $n = 27$ and $g = 3$), Theorem 5.5 applies to $13 \leq m \leq 15$. For $m = 13$, $n-2m+4g+q-2 = 14 = 4\cdot 3+2$, so that $u = 4$ and $v = 2$. Thus from Theorem 5.5, for $m = 13$, $\nabla_{11}(C_m) = \nabla(C_m)$ and $\nabla(C_m) = 10$. For $m = 14$, we have $u = 4$ and $v = 0$ so that Theorem 5.5 gives $\nabla_{12}(C_m) = \nabla(C_m)$ and $\nabla(C_m) = 11$. For $m = 15$, we have $u = 3$ and $v = 1$ so that Theorem 5.5 gives $\nabla_{13}(C_m) = \nabla(C_m)$ and $\nabla(C_m) = 11$. These are all in agreement with Example 5.3.2.

Table 1 shows the DLP bounds given by Theorem 5.5, for small values of q not covered in Examples 5.7.

Table 1: DLP Bounds for $q \in [4, 8]$ and $m \in J(n, g)$.

q															
4	m	32	33	34	35	36	37								
	$\nabla(C_m)$	26	27	27	28	27	28								
5	m	62	63	64	65	66	67	68	69	70	71				
	$\nabla(C_m)$	52	53	54	54	55	55	55	56	55	56				
7	m	171	172	173	174	175	176	177	178	179	180				
	$\nabla(C_m)$	150	151	152	153	153	154	155	155	155	156				
	m	181	182	183	184	185	186	187	188	189	190	191			
	$\nabla(C_m)$	157	156	157	158	158	157	158	159	158	158	159			
8	m	256	257	258	259	260	261	262	263	264	265	266	267	268	269
	$\nabla(C_m)$	228	229	230	231	231	232	233	234	233	234	235	236	235	236
	m	270	271	272	273	274	275	276	277	278	279	280	281	282	283
	$\nabla(C_m)$	237	238	237	237	238	239	238	238	239	240	239	238	239	240

We conclude by comparing the DLP bounds for Hermitian codes given in Examples 5.7 and Table 1 with the Clifford bound given by Corollary 3.5. We note that Hermitian function fields are not hyperelliptic (since if $g \geq 2$ then $q \geq 3$ and $\Gamma(2) = q \geq 3$). The Clifford bounds for $q \in [2, 8]$ and $m \in J(n, g)$ are given in Table 2. For $q \in \{2, 4, 5, 7, 8\}$, the Clifford bound is equal to the DLP bound for $m = \lfloor \frac{n}{2} \rfloor$ and inferior to it otherwise. For $q = 3$, the Clifford bound is equal to the DLP bound for $m \in \{13, 15\}$ and inferior to it for $m = 14$.

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Note added in proof. The state complexity of Hermitian codes has also been studied in Shany, Y. and Be'ery, Y: 'Bounds on the state complexity of codes from the Hermitian function field and its subfields', IEEE Trans. Info. Theory, 46, 1523–1527 (2000). Proposition 3.4 above is a sharper version of Proposition 1, *loc. cit.* and [4, Example 5.11] generalizes their main result to arbitrary self-dual Hermitian codes.

Table 2: Clifford Bounds for $q \in [2,8]$ and $m \in J(n,g)$.

q			
2	m	4	
	Clifford bound	3	
3	m	$[13,14]$	15
	Clifford bound	10	11
4	m	$[32,37]$	
	Clifford bound	26	
5	m	$[62,63]$	$[64,71]$
	Clifford bound	52	53
7	m	$[171,172]$	$[173,191]$
	Clifford bound	150	151
8	m	$[256,283]$	
	Clifford bound	228.	

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