Critical Sets In Latin Squares Of Order Less Than 11

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Abstract

A critical set C in a latin square L is a partial latin square which has a unique completion to L and for which no subset of C has this property. In this paper I document known results on the possible sizes of critical sets, and provide a reference list for the existence of critical sets in latin squares of order less than or equal to 10. Many of the results in this list are new and where this is the case I exhibit a critical set of the given size in the Appendix.

1 Introduction

A *latin square* L of order n is an $n \times n$ array with entries chosen from a set N, of size n, such that each element of N occurs precisely once in each row and column. A *partial latin square* P of order n is an $n \times n$ array with entries chosen from a set N, of size n, such that each element of N occurs at most once in each row and column. A partial latin square $C = \{(i,j;k) \mid \text{ cell } (i,j) \text{ contains } k\}$, of order n, is said to have a *unique completion* to the latin square L, if L is the only latin square of order n which has element k in position (i,j), for each $(i,j;k) \in C$. A *critical set*, in a latin square L of order n, is a set $C = \{(i,j;k) \mid \text{ cell } (i,j) \text{ contains } k\}$ such that,

1. C has a unique completion to L, and
2. no proper subset of C satisfies 1.

A critical set is a partial latin square which is contained in precisely one latin square, of the same order, with the additional property that if one removes any entry from the partial latin square, then what is left is contained in at least two latin squares of the same order. A latin square L and a critical set C in L are displayed below.

0	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
3	4	5	6	0	1	2
4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	0	1	2	3	4	5

L

0	1	2	3			
1	2	3				
2	3					
3						
						4
					4	5

C

The latin square and critical set are of order 7. The *size* of a critical set is the number of non-empty cells in the partial latin square. The size of the critical set given above is 13.

In this paper I discuss the existence of critical sets in latin squares, of order less than or equal to 10. In Section 2 of this paper, I document the known results on the size of critical sets in latin squares. For latin squares of order n, where $1 \leq n \leq 10$, I will list the size of the smallest known critical set and the size of the largest known critical set. Then for all possible sizes between these two bounds, with the exception of order 8, size 17 and order 10, sizes 26 and 28, I establish the existence of a critical set of the given size. These results are summarised in a table, in Section 3. If the existence of the critical set has been established using the results listed in Section 2, I note the appropriate theorem and reference the original work. In all other cases (56 in total) the existence of the critical set is established for the first time in this paper and examples of such critical sets are given in the Appendix.

2 Known results

Critical sets were first discussed by Nelder [**11**] in 1977. Let $lcs(n)$ denote the size of the largest critical set in any latin square of order n and scs denote the size of the smallest critical set in any latin square of order n. It is easily shown that for latin squares, of order n, $lcs(n) \leq n^2 - n$ and $scs(n) \geq n - 1$. Nelder conjectured that $lcs(n) = (n^2 - n)/2$ and $scs(n) = \lfloor n^2/4 \rfloor$. The bound for lcs was shown to be false in 1982. Stinson and van Rees, [**14**], exhibited examples of critical sets for which $lcs(n) \geq (n^2 - n)/2$. They also showed that if $\delta(n) = max|C|/n^2$, taken over all critical sets of L, then the $limsup\ \delta(n)$, as $n \to \infty$, is 1.

As for scs, to date, there appears to be no evidence which suggests that Nelder's conjecture is false. In 1994, it was shown that $scs(n) \geq n + 1$, [**2**], and recently Fu, Fu and Rodger showed that $scs(n) \geq \lceil \frac{3n}{2} \rceil - 2$.

An upper bound on $scs(n)$ was established by Curran and van Rees [**4**] in 1978 when they established the existence of critical sets of size $n^2/4$, when n is even, (see also Smetaniuk, [**13**]) and established the existence of a uniquely completable set of size $(n^2 - 1)/4$ when n is odd. Consequently, $scs(n) \leq n^2/4$ when n is even and $scs(n) \leq (n^2 - 1)/4$ when n is odd.

For particular values of n, Curran and van Rees [**4**] verified that when $n = 1, 2, 3, 4$ and 5, $scs(n) = 0, 1, 2, 4$ and 6 respectively. When $n = 1, 2$ and 3, they verified that $lcs(n) = 0, 1$ and 3 respectively. In 1982, Stinson and van Rees **[14]** extended this work and found lower bounds for $lcs(n)$ when $n = 2, \ldots, 10$. Their results are summarised in the following table.

Order n	Upper bound on $scs(n)$	Known bound on $lcs(n)$	Order n	Upper bound on $scs(n)$	Known bound on $lcs(n)$
1	0	0	2	1	1
3	2	3	4	4	7
5	6	10	6	9	18
7	12	24	8	16	37
9	20	39	10	25	55

I conclude this section with known results on the existence of critical sets of a given size.

Theorem 1 *(Curran and van Rees* **[4]***) For the addition table of the integers modulo n, where n is even, there exists critical sets of size*

$$\frac{n^2}{4}.$$

Theorem 2 *(Cooper, Donovan and Seberry* **[1]***) For the addition table of the integers modulo n, where n is odd, there exists critical sets of size*

$$\frac{n^2-1}{4}.$$

Theorem 3 *(Donovan and Cooper* **[6]***) For the addition table of the integers modulo n, there exists critical sets of size*

$$r^2 + 3r - nr + 2 + \frac{n^2 - 3n}{2}$$

where r is an integer such that $\frac{n-3}{2} \leq r \leq n-2$.

Howse and Sittampalam (with Keedwell) have obtained results relating to the Dihedral group.

Theorem 4 *(Howse* **[8]***) For the multiplication table of the Dihedral group, of order* $n = 2m$, *there exists critical sets of the following sizes:*

$$\frac{7m^2-2m}{4}, \text{ if } m \text{ is even, or } \frac{7m^2-2m-1}{4}, \text{ if } m \text{ is odd, and}$$
$$\frac{7m^2-6m+4}{4}, \text{ if } m \text{ is even, or } \frac{7m^2-6m+3}{4}, \text{ if } m \text{ is odd.}$$

Theorem 5 *(Sittampalam (with Keedwell),* **[12]***) For the multiplication table of the Dihedral group, of order $n = 2m$, there exists a critical set of size*

$$2m^2 - 3m + 3.$$

The following two results deal with to latin squares which are the direct product of the addition table for the integers modulo 2 and the addition table of the integers modulo m.

Theorem 6 *(Stinson and van Rees,* **[14]***) For the latin square representing the direct product of C_2, the cyclic group of order 2, with C_m, the cyclic group of order m, where m is even, there exists a critical set of size*

$$\frac{7m^2}{4}.$$

Proof. This result can be obtained by applying Theorem 2.8 of Stinson and van Rees's paper to Theorem 1 of this paper.

Theorem 7 *(Cooper, Donovan and Gower* **[3]***) For the latin square representing the direct product of C_2, the cyclic group of order 2, with C_m, the cyclic group of order m, where m is odd and greater than or equal to 3, there exists a critical set of size*

$$\frac{52m^2 - 20}{32}.$$

The next result establishes the existence of $m - 1$ critical sets, in latin squares of order $2m$. The critical set of size $2m^2 - 3m + 3$ corresponds to that given by Sittampalam (with Keedwell) **[12]**.

Theorem 8 *(Donovan* **[7]***) There exists critical sets of sizes*

$$2m^2 - 3m + 3, \ \ldots, \ 2m^2 - 2m + 1$$

in latin squares of order $n = 2m$.

In the next three results, for n large, the size of the critical set is greater than $\frac{n^2-n}{2}$, (the value Nelder original conjectured for $lcs(n)$).

Theorem 9 *(Donovan* **[7]***) There exists a critical set of size*

$$\frac{5m^2 - 3m}{2}$$

in a latin square of order $n = 2m$.

Theorem 10 *(Donovan* **[7]***) There exists a critical set of size*

$$\frac{17}{2}k^2 - \frac{9}{2}k + 1$$

in a latin square of order $n = 4k$.

Theorem 11 *(Donovan* [**7**]*) There exists a critical set of size*

$$10k^2 - 7k + 2$$

in a latin square of order $n = 4k$.

Theorem 12 *(Mortimer* [**10**]*) The following partial latin square is a critical set of order 6 and size 10.*

1					
2					*3*
3			*2*		*1*
	6			*1*	
	4	*6*			

3 Critical sets of order less than or equal to 10

In this section I verify the existence of critical sets in latin squares of order less than or equal to 10. These results are summarised in table form.

Here the numbers in column 1 represent, n, the order of the critical set in question. Column 2 gives the size of the smallest critical set, of order n, known to exist. So for n even it is $n^2/4$ and for n odd it is $(n^2-1)/4$. Column 3 lists the largest known critical set for any latin square of order n. These numbers have been taken from the Stinson and van Rees paper [**14**]. Column 4 gives all possible values between the known bounds on scs and lcs. Column 5 provides evidence of the existence of critical sets of the given size. If the existence of the critical set has been established in the known literature, then the appropriate theorem and original reference are given. If the existence of the critical set is established for the first time in this paper, the reader is referred to the appropriate example in the Appendix. These examples have been found using programs written by Ian Mortimer [**10**]. In the case of critical sets of order 8 and size 17 and order 10 and sizes 26 and 28, the existence of such critical sets is still in doubt. In such cases ?? appears in Column 5.

Order, n	scs	lcs	Size	Exists, see Refernece [**x**]
1	0	0		
			0	[**4**]
2	1	1		
			1	[**4**]
3	2	3		
			2	[**4**]
			3	[**4**]
4	4	7		
			4	Theorem 1, [**4**]
			5	[**5**]
			6	[**5**]
			7	See Lemma 3.2 of [**14**]
5	6	10		
			6	Theorem 1, [**1**]
			7	[**1**]
			8	See Appendix #1
			9	See Appendix #2
			10	See Lemma 3.3 of [**14**]
6	9	18		
			9	Theorem 1, [**4**]
			10	Theorem 12, [**10**]
			11	Theorem 3, [**6**]
			12	Theorem 5, [**12**]
			13	Theorem 8, [**7**]
			14	Theorem 7, [**7**]
			15	Theorem 3, [**6**]
			16	See Appendix #3
			17	See Appendix #4
			18	Lemma 3.4,[**14**], see also Theorem 9, [**7**]
7	12	24		
			12	Theorem 2, [**1**]
			13	Theorem 3, [**6**]
			14	See Appendix #5
			15	See Appendix #6
			16	Theorem 3, [**6**]
			17	See Appendix #7
			18	See Appendix #8
			19	See Appendix #9
			20	See Appendix #10

Order, n	scs	lcs	Size	Exists, see Refernece [**x**]
7 cont.			21	Theorem 3, [**6**]
			22	See Appendix #11
			23	See Appendix #12
			24	See Lemma 3.5 of [**14**]
8	16	37		
			16	Theorem 1, [**4**]
			17	??
			18	Theorem 3, [**6**]
			19	See Appendix #13
			20	See Appendix #14
			21	See Appendix #15
			22	Theorem 3, [**6**]
			23	Theorem 5, [**12**]
			24	Theorem 8, [**7**]
			25	Theorem 8, [**7**]
			26	Theorem 4, [**8**]
			27	See Appendix #16
			28	Theorem 6, [**14**]
			29	See Appendix #17
			30	See Appendix #18
			31	See Appendix #19
			32	See Appendix #20
			33	See Appendix #21
			34	Theorem 9, [**7**]
			35	See Appendix #22
			36	See Appendix #23
			37	Lemma 3.2 of [**14**], see also Theorem 9, [**7**]
9	20	39		
			20	Theorem 2, [**1**]
			21	Theorem 3, [**6**]
			22	See Appendix #24
			23	See Appendix #25
			24	Theorem 3, [**6**]
			25	See Appendix #26
			26	See Appendix #27
			27	See Appendix #28
			28	See Appendix #29
			29	Theorem 3, [**6**]
			30	See Appendix #30

Order, n	scs	lcs	Size	Exists, see Refernece [**x**]
9 cont.			31	See Appendix #31
			32	See Appendix #32
			33	See Appendix #33
			34	See Appendix #34
			35	See Appendix #35
			36	Theorem 3, [**6**]
			37	See Appendix #36
			38	See Appendix #37
			39	See Lemma 3.6 of [**14**]
10	25	55		
			25	Theorem 2, [**1**]
			26	??
			27	Theorem 3, [**6**]
			28	??
			29	See Appendix #38
			30	See Appendix #39
			31	Theorem 3, [**6**]
			32	See Appendix #40
			33	See Appendix #41
			34	See Appendix #42
			35	See Appendix #43
			36	See Appendix #44
			37	Theorem 3, [**6**]
			38	Theorem 5, [**12**]
			39	Theorem 8, [**7**]
			40	Theorem 7, [**3**]
			41	Theorem 4, [**8**]
			42	See Appendix #45
			43	See Appendix #46
			44	See Appendix #47
			45	Theorem 3, [**6**]
			46	See Appendix #48
			47	See Appendix #49
			48	See Appendix #50
			49	See Appendix #51
			50	See Appendix #52
			51	See Appendix #53
			52	See Appendix #54
			53	See Appendix #55
			54	See Appendix #56
			55	Lemma 3.7 of [**14**], see also Theorem 9, [**7**]

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Appendix

NUMBER 1

ORDER: 5 SIZE: 8

1	2			
2		4		
	4			
4				3
			3	

NUMBER 2

ORDER: 5 SIZE: 9

1	2	3		
2	3	4		
	4			
4				
			3	

NUMBER 3

ORDER: 6 SIZE: 16

		2	4	5	6
	1		6	4	
			5	6	4
	6	5		2	3
		6			2

NUMBER 4

ORDER: 6 SIZE: 17

	3	2	4	5	6
		3	6		5
			5	6	4
	6			2	3
	4	6			2

NUMBER 5

ORDER: 7 SIZE: 14

1	2		4			
2		4				
	4					
4						3
					3	
				3		5
			3		5	6

NUMBER 6

ORDER: 7 SIZE: 15

1	2	3		5		
2	3		5			
3		5				
	5					
5						4
					4	
				4		6

NUMBER 7

ORDER: 7 SIZE: 17

	2	3		5		
2	3		5			
3		5				
	5			1		
5						4
		1			4	
	1			4		6

NUMBER 8

ORDER: 7 SIZE: 18

1	2	3				
2	1					
3			6	7		
	5	6	7			1
		7			3	
				1		5
					5	6

NUMBER 9

ORDER: 7 SIZE: 19

1		3	4	5	6	
	3	4	5	6		
3	4					2
4	5	6				
5	6					
6						
		2				

NUMBER 10

ORDER: 7 SIZE: 20

1	2	3	4	5		
2	3	4	5	6		
3	4	5	6			
4	5	6				
	6					
6						
					5	

NUMBER 11

ORDER: 7 SIZE: 22

2	1				5	3
1	3	7				
		4	1			5
7		1	5		2	4
5			2		7	
3	2		4			1

NUMBER 12

ORDER: 7 SIZE: 23

	1	6		4		3
1	3					2
		4	1			5
7		1	5		2	4
5			2		7	
3	2	5	4			1

NUMBER 13

ORDER: 8 SIZE: 19

1	2	3					
2	3						
3							
			7				
		7					4
	7					4	5
7					4	5	6
				4	5	6	

NUMBER 14

ORDER: 8 SIZE: 20

1	2		4	5			
2		4	5				
	4	5					
4	5						3
5						3	
					3		
				3			6
			3			6	7

NUMBER 15

ORDER: 8 SIZE: 21

	2	3	4				
2	3	4					
3	4			7			
4			7				
		7		1			
			1				5
7						5	6
	1				5	6	

NUMBER 16

ORDER: 8 SIZE: 27

1	2	3	4	5	6		
2	3	4	5	6	7		
3	4	5	6	7			
4	5	6	7				
5	6	7					
	7						
7							
						6	

NUMBER 17

ORDER: 8 SIZE: 29

1	2			6		7	
2	1	4	3				
3		1		7		5	6
						6	5
5		7	8	1		3	
		8	7				
7		5		3	4	1	
				4	3		

NUMBER 18

ORDER: 8 SIZE: 30

1	2		4	5		7	
2	1	4	3				
3		1		7		5	6
						6	5
5		7	8	1		3	
		8	7				
7		5		3	4	1	
				4	3		

NUMBER 19

ORDER: 8 SIZE: 31

		3	2	5	6	7	8
			3	8	5	6	7
				7	8	5	6
4				6	7		5
	8	7	6		2	3	4
		8	7			2	3
			8				2

NUMBER 20

ORDER: 8 SIZE: 32

1	2	3	4	5	6		
2		4	3				7
	4	1	2		8		
4	3		1			6	
5	6	7		1	2	3	
6	5			2	1		
7		5		3		1	

NUMBER 21

ORDER: 8 SIZE: 33

		3	2	5	6	7	8
		4		8	5	6	7
			4	7	8	5	6
	3			6	7	8	5
	8	7	6		2	3	4
		8	7			2	3
			8				2

NUMBER 22

ORDER: 8 SIZE: 35

1	2	3	4	5	6		
2		4	3	6			7
3	4	1	2	7		5	
4	3	2	1				
5	6	7		1	2	3	
6	5			2	1		
7		5		3		1	

NUMBER 23

ORDER: 8 SIZE: 36

	4	3	2	5	6	7	8
		4	3	6	5	8	7
			4	7	8	5	6
				8	7	6	5
	6	7	8		2	3	4
		8	7			4	3
	8		6		4		2

NUMBER 24

ORDER: 9 SIZE: 22

1	2	3		5				
2	3		5					
3		5						
	5							
5								4
							4	
						4		6
					4		6	7
				4		6	7	8

NUMBER 25

ORDER: 9 SIZE: 23

1	2	3	4		6			
2	3	4		6				
3	4		6					
4		6						
	6							
6								5
							5	
						5		7
					5		7	8

NUMBER 26

ORDER: 9 SIZE: 25

1	2	3		5	6			
2	3		5	6				
3		5	6					
	5	6						
5	6							4
6							4	
						4		
					4			7
				4			7	8

NUMBER 27

ORDER: 9 SIZE: 26

1	2	3	4	5		7		
2	3	4	5		7			
3	4	5		7				
4	5		7					
5		7						
	7							
7								6
							6	
						6		8

NUMBER 28

ORDER: 9 SIZE: 27

1	2	3		5		7		
2	3		5		7			
3		5		7				
	5		7					
5		7						4
	7						4	
7						4		6
					4		6	
				4		6		8

NUMBER 29

ORDER: 9 SIZE: 28

1	2	3	4		6	7		
2	3	4		6	7			
3	4		6	7				
4		6	7					
	6	7						
6	7							5
7							5	
						5		
					5			8

NUMBER 30

ORDER: 9 SIZE: 30

1	2	3		5	6	7		
2	3		5	6	7			
3		5	6	7				
	5	6	7					
5	6	7						4
6	7						4	
7						4		
					4			
				4				8

NUMBER 31

ORDER: 9 SIZE: 31

1	2	3	4	5		7		
2	3	4	5		7			
3	4	5		7	8			
4	5		7	8				
5		7	8					
	7	8						
	8							6
8							6	
						6	7	

NUMBER 32

ORDER: 9 SIZE: 32

1	2	3	4	5	6			
2	3	4	5	6	7			
3	4		6	7				
4		6	7	8				
		7	8					
	7	8						
	8						5	
8						5	6	
					5	6	7	

NUMBER 33

ORDER: 9 SIZE: 33

1	2	3	4	5	6			
2	3	4	5	6	7	8		
3	4		6	7				
4		6	7	8				
	6	7	8					
6	7	8						
	8						5	
8						5		
					5		7	

NUMBER 34

ORDER: 9 SIZE: 34

1	2	3	4	5	6	7		
2	3	4	5	6	7			
3	4	5	6	7	8			
4	5	6	7	8				
5		7	8					
	7	8						
	8							
8							6	
						6	7	

NUMBER 35

ORDER: 9 SIZE: 35

1	2	3	4	5	6	7		
2	3	4	5	6	7	8		
3	4	5	6	7	8			
4	5	6	7	8				
5	6	7	8					
6	7	8						
	8							
8								
							7	

NUMBER 36

ORDER: 9 SIZE: 37

1	2	3	4	5		7	8	
2	3	1	5			8		
3	1							
4	5		7			1	2	
5						2		
					8			
7	8		1	2		4	5	6
8			2			5	6	4
						6	4	5

NUMBER 37

ORDER: 9 SIZE: 38

1	2	3	4			7	8	
2	3	1				8		
3	1	2			5			
4	5		7	8		1	2	
5			8			2		
7	8		1	2		4	5	6
8			2			5	6	4
						6	4	5

NUMBER 38

ORDER: 10 SIZE: 29

1	2	3		5	6				
2	3		5	6					
3		5	6						
	5	6							
5	6								4
6								4	
							4		
						4			7
					4			7	8
				4			7	8	9

NUMBER 39

ORDER: 10 SIZE: 30

1	2		4	5	6				
2		4	5	6					
	4	5	6						
4	5	6							
5	6							3	
6							3		
						3			
					3				7
				3				7	8
			3				7	8	9

NUMBER 40

ORDER: 10 SIZE: 32

	2	3	4	5					
2	3	4	5						
3	4	5							
4	5				9				
5				9					
			9		1				
		9		1					6
			1					6	7
9							6	7	8
	1					6	7	8	

NUMBER 41

ORDER: 10 SIZE: 33

1	2	3		5	6	7			
2	3		5	6	7				
3		5	6	7					
	5	6	7						
5	6	7							4
6	7							4	
7							4		
						4			
					4				8
				4				8	9

NUMBER 42

ORDER: 10 SIZE: 34

1	2		4	5					
2		4	5			8			
	4	5			8				
4	5			8					3
5			8					3	
		8					3		
	8					3			6
8					3			6	7
				3			6	7	
			3			6	7		9

NUMBER 43

ORDER: 10 SIZE: 35

1	2	3	4	5		7	8		
2	3	4	5		7	8			
3	4	5		7	8				
4	5		7	8					
5		7	8						
	7	8							
7	8								6
8								6	
							6		
						6			9

NUMBER 44

ORDER: 10 SIZE: 36

1		3		5					
	3		5						
3		5							2
	5				9			2	
5							2		4
			9			2		4	
					2		4		6
	9			2		4		6	7
9			2		4		6	7	8
		2		4		6	7	8	

NUMBER 45

ORDER: 10 SIZE: 42

1		3	4	5		7	8		
	3	4	5		7	8	9		
3	4	5		7	8				
4	5		7	8	9				
5		7	8	9			2		
	7	8	9			2			
7	8								6
8	9							6	
9			2				6		
		2				6			

NUMBER 46

ORDER: 10 SIZE: 43

1	2	3	4	5	6	7	8		
2	3	4	5	6	7	8			
3	4	5	6	7	8	9			
4	5	6	7	8	9				
5	6	7	8	9					
6		8	9						
	8	9							
8	9								
9								7	
							7		

NUMBER 47

ORDER: 10 SIZE: 44

1	2	3	4	5	6	7	8		
2	3	4	5	6	7	8	9		
3	4	5	6	7	8	9			
4	5	6	7	8	9				
5	6	7	8	9					
6	7	8	9						
7	8	9							
	9								
9									
								8	

NUMBER 48

ORDER: 10 SIZE: 46

			3	2	6	7	8		10
		5		3	10	6	7	8	
			5		9		6	7	8
4				5	8	9	10	6	7
	4				7	8	9	10	6
			8	7		2		4	5
		10		8			2		4
			10					2	
9				10	3				
	9					3			

NUMBER 49

ORDER: 10 SIZE: 47

		4	3	2	6	7	8	9	10
			4	3		6	7	8	9
				4	9	10	6	7	8
					8	9	10	6	7
5					7	8	9		6
	10	9	8	7			3	4	5
		10	9	8				3	4
			10	9					3
				10					
					2				

NUMBER 50

ORDER: 10 SIZE: 48

		4	3	2	6	7	8	9	10
			4	3	10	6	7	8	9
				4	9		6	7	8
					8	9	10	6	7
5					7	8	9		6
	10	9	8	7				4	5
		10	9	8			2		4
			10	9				2	
				10	3				2
						3			

NUMBER 51

ORDER: 10 SIZE: 49

		4	3	2	6	7	8	9	10
			4	3	10	6	7	8	9
				4	9		6	7	8
					8	9	10	6	7
5					7	8	9		6
	10	9	8	7			3	4	5
		10	9	8			2	3	4
			10	9				2	
				10					2
						3			

NUMBER 52

ORDER: 10 SIZE: 50

	5	4	3	2	6	7	8	9	10
		5	4	3	10	6	7	8	
			5	4		10	6	7	8
				5	8	9	10	6	7
					7	8	9	10	6
		9	8	7				4	5
		10	9	8			2		4
			10					2	
				10	3				2
	9					3			

NUMBER 53

ORDER: 10 SIZE: 51

		4	3	2	6	7	8	9	10
			4	3	10	6	7	8	9
				4	9	10	6	7	8
					8	9	10	6	7
5					7	8	9		6
	10	9	8	7		2	3	4	5
		10	9	8			2	3	4
			10	9				2	3
				10					2

NUMBER 54

ORDER: 10 SIZE: 52

			3	2	6	7	8	9	10
		5		3	10	6	7	8	9
			5		9	10	6	7	8
4				5	8	9		6	7
	4				7	8	9	10	6
	10	9	8	7		2	3	4	5
		10	9	8			2	3	4
			10	9				2	3
				10					2

NUMBER 55

ORDER: 10 SIZE: 53

	5	4	3	2	6	7	8	9	10
		5	4	3	10	6	7	8	
			5	4	9	10	6	7	8
				5	8	9	10	6	7
					7	8	9	10	6
		9	8	7		2	3	4	5
		10	9	8			2	3	4
			10					2	3
				10					2
	9								

NUMBER 56

ORDER: 10 SIZE: 54

		4	3	2	6	7	8	9	10
		5	4	3	10	6	7	8	9
			5		9	10	6	7	8
				5	8	9	10	6	7
	4				7	8	9	10	6
	10	9	8	7		2	3	4	5
		10	9	8			2	3	4
			10	9				2	3
				10					2

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