Computing Closed Essential Surfaces in Knot Complements[*]

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ABSTRACT

We present a new, practical algorithm to test whether a knot complement contains a closed essential surface. This property has important theoretical and algorithmic consequences. However systematically testing it has until now been infeasibly slow, and current techniques only apply to specific families of knots. As a testament to its practicality, we run the algorithm over a comprehensive body of 2979 knots, including the two 20-crossing dodecahedral knots, yielding results that were not previously known.

The algorithm derives from the original Jaco-Oertel framework, involves both enumeration and optimisation procedures, and combines several techniques from normal surface theory. This represents substantial progress in the practical implementation of normal surface theory. Problems of this kind have a doubly-exponential time complexity; nevertheless, with our new algorithm we are able to solve it for a large and comprehensive class of inputs. Our methods are relevant for other difficult computational problems in 3-manifold theory, ranging from testing for Haken-ness to the recognition problem for knots, links and 3-manifolds.

Categories and Subject Descriptors

F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems—*Geometrical problems and computations*; G.4 [**Mathematical Software**]: Algorithm design and analysis

General Terms

Algorithms, theory, experimentation

Keywords

Computational topology, knot theory, incompressible surface, essential surface, normal surface

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1. INTRODUCTION

In the study of 3–manifolds, essential surfaces have been of central importance since Haken's seminal work in the 1960s. An essential surface may be regarded as 'topologically minimal', and there has since been extensive research into 3-manifolds, called *Haken 3-manifolds*, that contain an essential surface. The existence of such a surface has profound consequences for both the topology and geometry of a 3–manifold [13, 15, 20, 21, 22, 24, 27, 28, 31].

Given any closed 3-manifold, specified by a triangulation, it is a theorem of Jaco and Oertel [17] from 1984 that one may algorithmically test for the existence of a closed essential surface. However, their algorithm has significant intricacies and is of doubly-exponential complexity in terms of the input size, putting it well beyond the scope of a practical implementation.

In this paper we present for the first time a *practical* algorithm that, though still doubly-exponential in theory, is able to systematically test a significant class of 3-manifolds for the existence of a closed essential surface, and is both efficient in practice and always conclusive. To illustrate its power, we run this algorithm over a comprehensive body of input data, yielding computer proofs of new mathematical results.

The 3-manifolds we examine in this paper are the motivating spaces for 3-manifold theory: knot complements. These are the spaces that arise by removing a knotted curve from 3-dimensional space, although our methods can be extended to apply to a far wider class of 3-manifolds. See the full version of this paper for generalisations. In this paper we work with two collections of input data. First, for each of the 2977 non-trivial prime knots that can be drawn with a diagram of at most 12 crossings, we determine whether its complement contains a closed essential surface. If there is no such surface, the knot is called *small*, otherwise we call it *large*. Second, we apply our algorithm to resolve, in the affirmative, a question of Michel Boileau [4] who enquired whether two special 20-crossing knots called *dodecahedral knots* contain a closed essential surface in their complements. This question was recently also independently resolved by Jessica Banks [3] using non-computational techniques.

The algorithm presented here is theoretically significant because it is the first algorithm in the literature for testing largeness of arbitrary knots. However, more important is its practical significance: this is the first conclusive algorithm of this type that is implemented and fast enough for real-world use. The prior state-of-the-art algorithm for detecting essential surfaces was used to prove that the Weber-Seifert dodecahedral space is non-Haken [11]; however, although it resolved a long-standing open problem, this prior algorithm relies on heuristic methods that only work for certain triangulations, and are only conclusive if no essential surface exists. In contrast, the algorithm described here can work with arbitrary triangulations of knot complements, and is found to be effective regardless of the final result.

Our methods can be applied to related invariants of knots and 3-manifolds. For instance, the smallest genus g of a closed essential surface is an important knot invariant about which little is known for the case $g \geq 2$, and our algorithm opens the door to formulating and testing new hypotheses. These methods may also be extended to test a wide variety of 3-manifolds for Haken-ness and related properties. More broadly, iterated exponential complexity algorithms arise frequently in 3-manifold theory, and our methods give an outline for how such problems, like the recognition problem for knots and 3-manifolds, may one day be within the realm of a practical implementation.

We base our work on the framework of the Jaco-Oertel algorithm for testing for closed incompressible surfaces. This uses *normal surfaces*, which allow us to translate topological questions about surfaces into the setting of integer and linear programming. The framework consists of two stages: the first constructs a finite list of candidate essential surfaces, and the second tests each surface in the list to see if it is essential. A key difficulty with this framework, which our algorithm also inherits, is that both stages have running times that are worst-case exponential in their respective input sizes. Moreover, the output of the first stage is exponential in its input, and this then becomes the input to the second stage. This means that combining the two stages in any obvious way leads to a doubly-exponential time complexity solution.

Despite this significant hurdle, we introduce several innovations that cut down the running time enormously for both stages. Our optimisation for the first stage involves a combination of established techniques that, though well understood individually, require new ideas and theory in order to work harmoniously together. For the second stage we combine branch-and-bound techniques from integer programming with the Jaco-Rubinstein procedure for crushing surfaces within triangulations, extending recent work of the first author and Ozlen [9]. In more detail:

- For the first stage (enumerating candidate essential surfaces), we combine several techniques. First, we wish to create a triangulation for each knot complement that contains as few tetrahedra as possible. If one uses classical triangulations one needs as many as 50 tetrahedra for some knots in the 12-crossing tables, a size for which enumerating candidate surfaces is thoroughly infeasible even for modern high-performance machines. We therefore use *ideal triangulations* for knot complements, which are decompositions of these spaces into tetrahedra with their vertices removed. These introduce some significant theoretical difficulties, but they are much smaller with roughly half as many tetrahedra.

  Second, we use a variant of normal surface theory based on *quadrilateral coordinates*. The appeal is that this brings the dimension of the underlying integer and linear programming problems down from $7t$ in the classical setting to $3t$, where t is the number tetrahedra in the input 3-manifold. These coordinates were known to Thurston and Jaco in the 1980s, and first appeared in print in work of Tollefson [30].

  A theoretical difficulty arises when combining ideal triangulations with quadrilateral coordinates: this introduces objects called *spun-normal surfaces*, which are properly embedded non-compact surfaces (essentially built from infinitely many pieces). We counter this by introducing extra linear constraints called *boundary equations* which, with the development of appropriate theory, restrict the solution space in question to closed surfaces only. In particular, using an extension of the work of Jaco and Oertel [17] from compact manifolds to non-compact manifolds by Kang [23], we show in Theorem 2 that for each manifold under consideration, there is a finite, constructible set of normal surfaces with the property that if the manifold in question contains a closed essential surface, then one must exist in this set.

- For the second stage (testing whether a candidate surface is essential), the Jaco-Oertel approach cuts along each candidate surface and inspects the boundary of the resulting 3-manifold to see if it admits a *compression disc* (such a disc certifies that a surface is non-essential). The key difficulty is that one requires a new triangulation for the cut-open 3-manifold: since the candidate surface may be very complicated, any natural scheme for cutting and re-triangulating yields a new triangulation with exponentially many tetrahedra in the worst case, taking us far beyond the realm in which normal surface theory has traditionally been feasible in practice. Since these new triangulations are the input for stage two, which is itself exponential time, we now see where the double exponential arises, and why the Jaco-Oertel framework has long been considered far from practical.

  We resolve this significant problem using a blend of techniques. First, we use strong simplification heuristics to reduce the number of tetrahedra. Next, we replace the traditional (and very expensive) enumeration-based search for compression discs with an *optimisation* process that maximises Euler characteristic. This uses the branch-and-bound techniques of [9], and allows us to quickly focus on a single candidate compression disc. We employ the crushing techniques of Jaco and Rubinstein [18] to quickly test whether this is indeed a compression disc, and (crucially) to reduce the size of the triangulation if it is not.

More generally, this issue of iterated-exponential complexity, coming from cutting and re-triangulating, arises with ubiquity when considering objects called *normal hierarchies*. These hierarchies are key when solving more difficult problems such as the recognition problem for knots and 3-manifolds. Our approach here is both fast in practice and always correct and conclusive, making it a substantial breakthrough that indicates that, despite their iterated-exponential time complexities, practical implementations of these more difficult algorithms might indeed be possible.

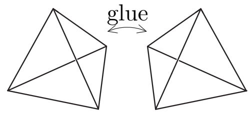


Figure 1: A 3-manifold may be specified by a triangulation.

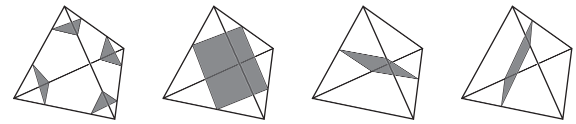


Figure 2: The seven types of normal disc in a tetrahedron.

2. PRELIMINARIES

2.1 Knots, surfaces and triangulations

A 3-manifold is a mathematical object that locally looks like 3-dimensional Euclidean space (or, for points on the boundary, 3-dimensional Euclidean half-space). Because every topological 3-manifold admits precisely one piecewise-linear structure (up to PL-homeomorphism) [25], in practice this means that 3-manifolds may be studied via triangulations. A *triangulation* of a compact 3-manifold M is a description of M as the disjoint union of a finite collection of 3-simplices with their faces identified in pairs, as shown in Figure 1.

A triangulation $\mathcal{T}$ for a 3-manifold M gives rise to *vertices*, *edges*, *faces* and *tetrahedra* in M. Edges whose interior lies in the interior of M are called *interior edges*, and edges that lie entirely on the boundary of M are called *boundary edges*. In practice, a tetrahedron in M might not be embedded; for instance, we even allow two faces of a tetrahedron to be identified in M. For a precise description of our set-up and a detailed example, please see the full version of this paper.

Such a triangulation can only specify a compact 3-manifold. However, we can triangulate a non-compact 3-manifold by deleting the vertices from each tetrahedron (i.e., instead of identifying the faces of tetrahedra, we identify the faces of a finite collection of tetrahedra minus their vertices). This constitutes an *ideal triangulation* for the resulting non-compact quotient space.

The *link* of a vertex V in a triangulation is the frontier of a small regular neighbourhood of V. In a triangulation of a compact 3-manifold, every vertex link is a sphere or a disc; in an ideal triangulation, vertex links can be surfaces of arbitrary genus.

The 3–manifolds we study in this paper are *knot complements*. These are 3-manifolds obtained by removing a *knot*, which is knotted closed curve in $\mathbb{R}^3$, from 3–dimensional Euclidean space. In practice it is convenient to compactify $\mathbb{R}^3$ with a point at infinity, yielding a compact 3-manifold called the *3-sphere*, denoted S^3. One then removes the knot from S^3 instead. For a knot K we call the resulting non-compact 3-manifold $S^3\backslash K$ the *complement of* K. Knot complements always have ideal triangulations [29, Proposition 1.2].

If instead we remove from S^3 a small open neighbourhood of a knot K we obtain a compact 3-manifold called the *exterior of* K. Since they are compact, knot exteriors may be specified by triangulations. There are well established techniques for translating between an ideal triangulation for a knot complement and a triangulation for the corresponding knot exterior.

A knot K is called *non-trivial* if it is not the boundary of an embedded disc in S^3.

In this paper we are interested in *closed essential surfaces* in knot complements. We define these now. Let K be a knot, and let M be the complement of K. A connected two-sided closed surface with positive genus S, embedded in M, is a *closed essential surface in* M if the following properties hold: (i) the surface S is *incompressible* (as defined below); and (ii) the surface S is not *boundary parallel*, that is, not ambient isotopic to a small tube running around K.

The definition of incompressible is as follows. A *compression disc* for an embedded surface S in a 3-manifold M is an embedded disc $D \subset M$ for which (i) $D \cap S$ equals the boundary of D (denoted ∂D); and (ii) ∂D is a non-trivial curve in S (meaning ∂D does not bound a disc in S). If the surface S admits a compression disc, then we say S is *compressible*, otherwise S is *incompressible*. An equivalent, algebraic criterion can be found in the full version of this paper.

2.2 Quadrilateral coordinates and Q–matching equations

We use normal surface theory to search for essential surfaces. A *normal surface* in a (possibly ideal) triangulation $\mathcal{T}$ is a properly embedded surface which intersects each tetrahedron of $\mathcal{T}$ in a disjoint collection of *triangles* and *quadrilaterals*, as shown in Figure 2. These triangles and quadrilaterals are called *normal discs*. In an ideal triangulation of a non-compact 3–manifold, a normal surface may contain infinitely many triangles; such a surface is called *spun-normal* [29]. A normal surface may be disconnected or empty.

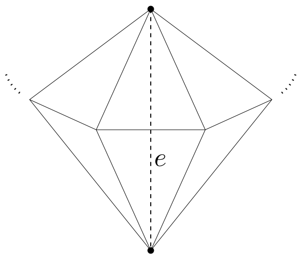
We now describe an algebraic approach to normal surfaces. The key observation is that each normal surface contains finitely many quadrilateral discs, and is uniquely determined (up to normal isotopy) by these quadrilateral discs. Here a *normal isotopy* of M is an isotopy that keeps all simplices of all dimensions fixed. Let $\square$ denote the set of all normal isotopy classes of normal quadrilateral discs in $\mathcal{T}$, so that $|\square| = 3t$ where t is the number of tetrahedra in $\mathcal{T}$. These normal isotopy classes are called quadrilateral *types*.

We identify $\mathbb{R}^\square$ with $\mathbb{R}^{3t}$. Given a normal surface S, let $x(S) \in \mathbb{R}^\square = \mathbb{R}^{3t}$ denote the integer vector for which each coordinate $x(S)(q)$ counts the number of quadrilateral discs in S of type $q \in \square$. This *normal Q–coordinate* $x(S)$ satisfies the following two algebraic conditions.

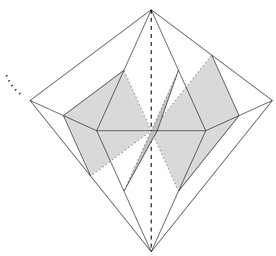
First, $x(S)$ is admissible. A vector $x \in \mathbb{R}^\square$ is *admissible* if $x \geq 0$, and for each tetrahedron x is non-zero on at most one of its three quadrilateral types. This reflects the fact that an embedded surface cannot contain two different types of quadrilateral in the same tetrahedron.

Second, $x(S)$ satisfies a linear equation for each interior edge in M, termed a *Q–matching equation*. Intuitively, these equations arise from the fact that as one circumnavigates the earth, one crosses the equator from north to south as often as one crosses it from south to north. We now give the precise form of these equations. To simplify the discussion, we assume that M is oriented and all tetrahedra are given the induced orientation; see [29, Section 2.9] for details.

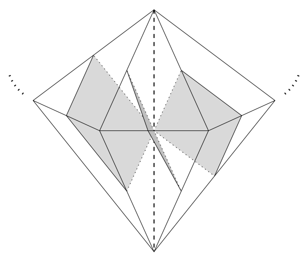
Consider the collection $\mathcal{C}$ of all (ideal) tetrahedra meeting



(a) The abstract neighbourhood $B(e)$



(b) Positive slope $+1$



(c) Negative slope -1

Figure 3: Slopes of quadrilaterals

at an edge e in M (including k copies of tetrahedron σ if e occurs k times as an edge in σ). We form the *abstract neighbourhood* $B(e)$ of e by pairwise identifying faces of tetrahedra in $\mathcal{C}$ such that there is a well defined quotient map from $B(e)$ to the neighbourhood of e in M; see Figure 3(a) for an illustration. Then $B(e)$ is a ball (possibly with finitely many points missing on its boundary). We think of the (ideal) endpoints of e as the poles of its boundary sphere, and the remaining points as positioned on the equator.

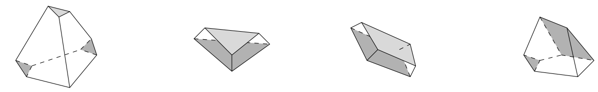
Let σ be a tetrahedron in $\mathcal{C}$. The boundary square of a normal quadrilateral of type q in σ meets the equator of $\partial B(e)$ if and only it has a vertex on e. In this case, it has a slope ± 1 of a well–defined sign on $\partial B(e)$ which is independent of the orientation of e. Refer to Figures 3(b) and 3(c), which show quadrilaterals with *positive* and *negative slopes* respectively.

Given a quadrilateral type q and an edge e, there is a *total weight* $\mathrm{wt}_e(q)$ of q at e, which records the sum of all slopes of q at e (we sum because q might meet e more than once, if e appears as multiple edges of the same tetrahedron). If q has no corner on e, then we set $\mathrm{wt}_e(q) = 0$. Given edge e in M, the Q–matching equation of e is then defined by $0 = \sum_{q\in\square} \mathrm{wt}_e(q)\ x(q)$.

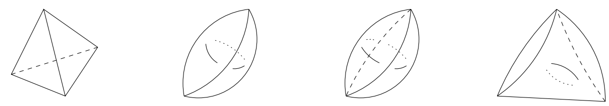
Theorem 1. *For each $x \in \mathbb{R}^\square$ with the properties that x has integral coordinates, x is admissible and x satisfies the Q–matching equations, there is a (possibly non-compact) normal surface S such that $x = x(S)$. Moreover, S is unique up to normal isotopy and adding or removing vertex linking surfaces, i.e., normal surfaces consisting entirely of normal triangles.*

This is related to Hauptsatz 2 of [14]. For a proof of Theorem 1, see [23, Theorem 2.1] or [29, Theorem 2.4].

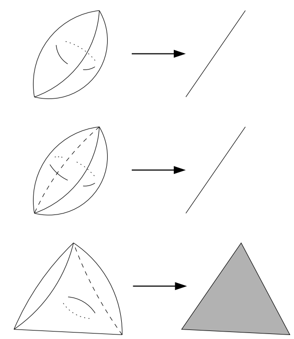
The set of all $x \in \mathbb{R}^\square$ with the property that (i) $x \geq 0$ and (ii) x satisfies the Q–matching equations is denoted $Q(\mathcal{T})$. This naturally is a polyhedral cone. Note however that the set of all admissible $x \in \mathbb{R}^\square$ typically meets $Q(\mathcal{T})$ in a non-convex set.



(a) Pieces after cutting open along S



(b) Pieces after crushing S to a point



(c) Flattening footballs and pillows

Figure 4: Steps in the Jaco-Rubinstein crushing process

2.3 Crushing triangulations

The crushing process of Jaco and Rubinstein [18] plays an important role in our algorithms, and we informally outline this process here. We refer the reader to [18] for the formal details, or to [7] for a simplified approach.

Let S be a two-sided normal surface in a triangulation $\mathcal{T}$ of a compact orientable 3-manifold M (with or without boundary). To *crush* S in $\mathcal{T}$, we (i) cut $\mathcal{T}$ open along S, which splits each tetrahedron into a number of (typically non-tetrahedral) pieces, several of which are illustrated in Figure 4(a); (ii) crush each resulting copy of S on the boundary to a point, which converts these pieces into tetrahedra, footballs and/or pillows as shown in Figure 4(b); and (iii) flatten each football or pillow to an edge or triangle respectively, as shown in Figure 4(c).

The result is a new collection of tetrahedra with a new set of face identifications. We emphasise that we *only* keep track of face identifications between tetrahedra: any "pinched" edges or vertices fall apart, and any lower-dimensional pieces (triangles, edges or vertices) that do not belong to any tetrahedra simply disappear. The resulting structure might not represent a 3-manifold triangulation, and even if it does the flattening operations might have changed the underlying 3-manifold in ways that we did not intend.

Although crushing can cause a myriad of problems in general, Jaco and Rubinstein show that in some cases the operation behaves extremely well [18]. In particular, if S is a normal sphere or disc, then after crushing we always obtain a triangulation of some 3-manifold M' (possibly disconnected, and possibly empty) that is obtained from the original M by zero or more of the following operations:

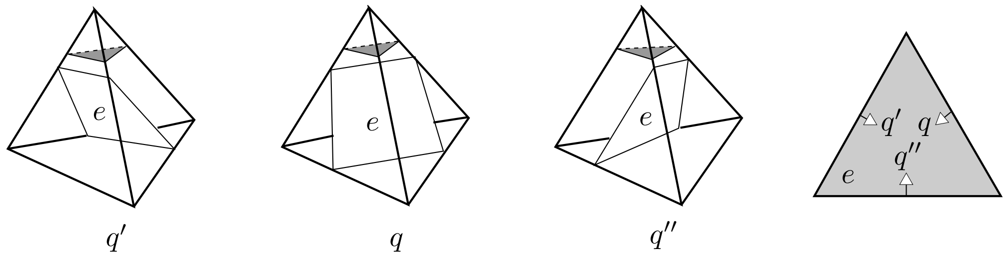
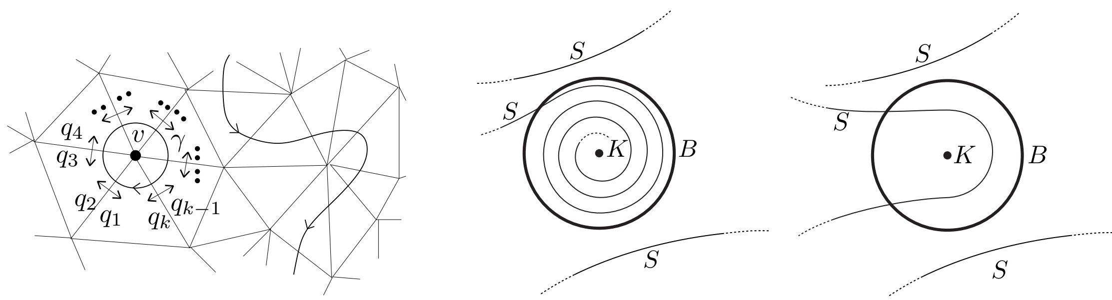


Figure 5: Coming up and dropping down



(a) $0 = \nu_x(\gamma) = \sum_{i=1}^{k}(-1)^i x(q_i)$ is the Q–matching equation

(b) spun $\iff \nu_x \neq 0$

(c) not spun $\iff \nu_x = 0$

Figure 6: Boundary map determines Q–matching equations and spinning

- cutting manifolds open along spheres and filling the resulting boundary spheres with 3-balls;
- cutting manifolds open along properly embedded discs;
- capping boundary spheres of manifolds with 3-balls;
- deleting entire connected components that are any of the 3-ball, the 3-sphere, projective space $\mathbb{R}P^3$, the lens space $L(3,1)$ or the product space $S^2 \times S^1$.

An important observation is that the number of tetrahedra that remain after crushing is precisely the number of tetrahedra that do not contain quadrilaterals of S.

3. CLOSED NORMAL SURFACES IN Q–SPACE

In this section we introduce the linear *boundary equations*, with which we restrict the normal surface solution space to closed surfaces only.

Let our knot complement be $M = S^3 \setminus K$. The ideal triangulation $\mathcal{T}$ of M has one ideal vertex, and its link is a torus. We view this torus T as made up of normal triangles, one near each corner of each ideal tetrahedron. Let $\gamma \in H_1(T;\mathbb{R})$. We now describe an associated linear functional $\nu(\gamma)\colon \mathbb{R}^\square \to \mathbb{R}$, which measures the behaviour along γ of a normal surface near the ideal vertex. The idea is similar to the intuitive description of the Q–matching equations. As one goes along γ and looks down into the manifold, normal quadrilaterals will (as Jeff Weeks puts it) *come up from below* or *drop down out of sight*. If the total number coming up minus the total number dropping down is non-zero, then the surface spirals towards the knot in the cross section $\gamma \times [0,\infty) \subset T \times [0,\infty)$ and the sign indicates the direction, see Figure 6(b). If this number is zero, then after a suitable isotopy the surface meets the cross section in a (possibly empty or infinite) union of circles, see Figure 6(c).

The torus T has an induced triangulation consisting of normal triangles. Represent γ by an oriented path on T, which is disjoint from the 0–skeleton and meets the 1–skeleton transversely. Each edge of a triangle in T is a normal arc. Give the edges of each triangle in T transverse orientations pointing into the triangle and labelled by the quadrilateral types sharing the normal arc with the triangle; see Figure 5. We then define $\nu(\gamma)$ as follows. Choosing any starting point on γ, we read off a formal linear combination of quadrilateral types q by taking $+q$ each time the corresponding edge is crossed with the transverse orientation, and $-q$ each time it is crossed against the transverse orientation (where each edge in T is counted twice—using the two adjacent triangles).

Evaluating $\nu(\gamma)$ at some $x \in \mathbb{R}^\square$ gives a real number $\nu_x(\gamma)$. For example, taking a small loop around a vertex in T and setting this equal to zero gives the Q–matching equation of the corresponding edge in M; see Figure 6(a). For each $x \in Q(\mathcal{T})$, the resulting map $\nu_x\colon H_1(T;\mathbb{R}) \to \mathbb{R}$ is a well-defined homomorphism, which has the property that the surface in Theorem 1 is closed if and only if $\nu_x = 0$ (see [29], Proposition 3.3). Since $\nu_x\colon H_1(T;\mathbb{R}) \to \mathbb{R}$ is a homomorphism, it is trivial if and only if we have $\nu_x(\alpha) = 0 = \nu_x(\beta)$ for any basis $\{\alpha, \beta\}$ of $H_1(T;\mathbb{R})$.

We define $Q_0(\mathcal{T}) = Q(\mathcal{T}) \cap \{x \mid \nu_x = 0\}$, and call a two-sided, connected normal surface F with $x(F)$ on an extremal ray of $Q_0(\mathcal{T})$ a Q_0*–vertex surface.* The following result is based on the seminal work of Jaco and Oertel [17]:

THEOREM 2. *Suppose M is the complement of a non-trivial knot in S^3. If M contains a closed essential surface*

S, then there is a normal, closed essential surface F with the property that $x(F)$ lies on an extremal ray of $Q_0(\mathcal{T})$. Moreover, if $\chi(S) < 0$, then there is such F with $\chi(F) < 0$.

PROOF (SKETCH). A complete proof of a more general statement is given in the full version of this paper. The key ideas are as follows. Given a closed essential surface in M, a standard argument shows that there is a *normal* closed essential surface in M. Amongst all normal surfaces isotopic (but not necessarily normally isotopic) to this, choose one that has minimal number of intersections with the 1–skeleton of the triangulation $\mathcal{T}$ (this is the PL analogue of a minimal surface). Denote this surface S.

If S is not a vertex surface, one can write it using a so-called *Haken sum* of vertex surfaces, which is a geometric realisation of the sum of Q–coordinate vectors. However, a complication arises, since only a multiple of S is known to be a Haken sum of vertex surfaces, and only up to vertex linking tori; that is, we only know that $nS + \Sigma = \sum n_i V_i = V + W$ for some $n \in \mathbb{N}$, where V is a vertex surface, Σ is vertex linking, and all other terms of the Haken sum are combined into the surface W. Building on Jaco and Oertel [17], Kang [23, Theorem 5.4] shows that V is incompressible for any such decomposition. Since Euler characteristic is additive under Haken sum, the result follows if $\chi(S) < 0$. If $\chi(S) = 0$, additional work is required to show that an essential torus cannot be written as a Haken sum of boundary parallel tori. □

4. ALGORITHMS

Here we describe the new algorithm to test whether a knot is large or small (i.e., whether its complement contains a closed essential surface). In this extended abstract we restrict our attention to the common setting of knots in the 3-sphere S^3. See the full version of this paper for an extension to the more general setting of links in arbitrary closed orientable 3-manifolds, as well as searching for essential surfaces in arbitrary closed orientable 3-manifolds (without knots or links).

We present the algorithm in two stages below. Algorithm 3 describes a subroutine to test whether a given closed surface is incompressible. Algorithm 5 is the main algorithm: it uses the results of Section 3 to identify candidate essential surfaces, and runs Algorithm 3 over each.

These algorithms contain a number of high-level and often intricate procedures, many of which are described in separate papers. For each algorithm, we discuss these procedures in further detail after presenting the overall algorithm structure.

ALGORITHM 3 (TESTING FOR INCOMPRESSIBILITY). *Let $\mathcal{T}$ be an ideal triangulation of a non-compact 3-manifold M that is the complement of a non-trivial knot in S^3. Let S be a closed two-sided normal surface of genus $g \geq 1$ within $\mathcal{T}$. To test whether S is incompressible in M:*

1. *Truncate the ideal vertex of $\mathcal{T}$ to obtain a compact manifold with boundary, cut $\mathcal{T}$ open along the surface S, and retriangulate.[1] The result is a pair of triangulations $\mathcal{T}_1, \mathcal{T}_2$ representing two compact manifolds with boundary M_1, M_2 (one on each side of S in M).*

   Let B_1, B_2 be the genus g boundary components of $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively that correspond to the surface S. Without loss of generality, suppose that the truncated ideal vertex was on the side of M_2; therefore $\mathcal{T}_2$ has an additional boundary torus, which we denote B_v.

2. *For each $i = 1, 2$:*

   (a) Simplify $\mathcal{T}_i$ into a triangulation with no internal vertices and only one vertex on each boundary component, without increasing the number of tetrahedra. Let the resulting number of tetrahedra in $\mathcal{T}_i$ be n.

   (b) Search for a connected normal surface E in $\mathcal{T}_i$ that is not a vertex link, has positive Euler characteristic, and (for the case $i = 2$) does not meet the torus boundary B_v.

   (c) If no such E exists, then there is no compressing disc for S in M_i. If $i = 1$ then try $i = 2$ instead, and if $i = 2$ then terminate with the result that S is incompressible.

   (d) Otherwise, crush the surface E as explained in Section 2.3 to obtain a new triangulation $\mathcal{T}_i'$ (possibly disconnected, or possibly empty) with strictly fewer than n tetrahedra. If some component of $\mathcal{T}_i'$ has the same genus boundary (or boundaries) as $\mathcal{T}_i$ then it represents the same manifold M_i, and we return to step 2a using this component of $\mathcal{T}_i'$ instead. Otherwise we terminate with the result that S is not incompressible.

Regarding the individual steps of this algorithm:

- Step 1 requires us to truncate an ideal vertex and cut a triangulation open along a normal surface. These are standard (though intricate) procedures. To truncate a vertex we subdivide tetrahedra and then remove the immediate neighbourhood of the vertex. To cut along a normal surface is more complex; a manageable implementation is described in [11].

- Step 2a requires us to simplify a triangulation to use the fewest possible vertices, without an increase in the number of tetrahedra. For this we begin with the rich polynomial-time simplification heuristics described in [6]. In practice, for all 2979 knots that we consider in Section 5, this is sufficient to reduce the triangulation to the desired number of vertices.

  If there are still extraneous vertices, we can remove these using the crushing techniques of Jaco and Rubinstein [18, Section 5.2]. This might fail, but only if ∂M_i has a compressing disc, or two boundary components of M_i are separated by a sphere; both cases immediately certify that the surface S is compressible, and we can terminate immediately.

- Step 2b requires us to locate a connected normal surface E in $\mathcal{T}_i$ that is not a vertex link, has positive Euler characteristic, and does not meet the torus boundary B_v. For this we use the recent method of [9, Algorithm 11], which draws on combinatorial optimisation techniques: in essence we run a sequence of linear programs over a combinatorial search tree, and prune this

[1] To *truncate* a vertex means to remove a small open neighbourhood of that vertex. Because the original ideal vertex has torus link, the truncated triangulation will acquire an additional torus boundary component.

tree using tailored branch-and-bound methods. See [9] for details.

We note that this search is the bottleneck of Algorithm 3: the search is worst-case exponential time, though in practice it often runs much faster [9]. The exposition in [9] works in the setting where the underlying triangulation is a knot complement, but the methods work equally well in our setting here. To avoid the boundary component B_v, we simply remove all normal discs that touch B_v from our coordinate system.

THEOREM 4. *Algorithm 3 terminates, and its output is correct.*

PROOF. The algorithm terminates because each time we loop back to step 2a we have fewer tetrahedra than before. Correctness is more interesting: there are many claims in the algorithm statement that require proof. Full proofs are given in the full version of this paper; the key ideas are as follows.

- In step 1 we claim that cutting along S yields two (disconnected) compact manifolds. This follows from the fact that every closed surface embedded in the 3-sphere is separating.
- In step 2c we claim that, if the surface E cannot be found in $\mathcal{T}_1$ *and* it cannot be found in $\mathcal{T}_2$, then the original surface S must be incompressible. This is because otherwise, by results of Jaco and Oertel [17], there must be a *normal* compressing disc on one side of S.
- In step 2d we make several claims. First, the new triangulation $\mathcal{T}_i'$ has strictly fewer tetrahedra because E is not a vertex link. Second, we claim that if $\mathcal{T}_i'$ has a component with the same genus boundary (or boundaries) as $\mathcal{T}_i$ then it represents the same manifold M_i, and otherwise S is compressible; this is because the "destructive" side-effects of the crushing process reduce the boundary genus by cutting along compressing discs for S.

There are additional complications involving irreducibility; again see the full version of this paper for details. □

ALGORITHM 5 (TESTING WHETHER A KNOT IS LARGE). *Let K be a non-trivial knot in S^3. To test whether K is large or small:*

1. *Build an ideal triangulation $\mathcal{T}$ of the complement of K in S^3.*
2. *Enumerate all extremal rays of $Q_0(\mathcal{T})$; denote these $\mathbf{e}_1, \ldots, \mathbf{e}_k$. For each extreme ray $\mathbf{e}_i$, let S_i be the unique connected two-sided normal surface for which $x(S_i)$ lies on $\mathbf{e}_i$. Ignore all surfaces S_i that are spheres.*
3. *For each remaining surface S_i, use algorithm 3 to test whether S_i is incompressible in $\mathcal{T}$. If any S_i is incompressible and is not a torus, then terminate with the result that K is large. If no S_i is incompressible, then terminate with the result that K is small.*
4. *Otherwise the only incompressible surfaces in our list are tori. For each incompressible torus S_i, test whether S_i is boundary parallel by (i) cutting $\mathcal{T}$ open along S_i, and then (ii) using the Jaco-Tollefson algorithm [19, Algorithm 9.7] to test whether one of the resulting components is the product space (Torus) $\times [0,1]$. If all incompressible tori are found to be boundary parallel then K is small, and otherwise K is large.*

Regarding the individual steps:

- Step 1 requires us to triangulate the complement of K. Hass et al. [16] show how to build a compact triangulation (with boundary triangles). To make this an ideal triangulation we cone over the boundary, and retriangulate to remove internal (non-ideal) vertices.
- Step 2 requires us to enumerate all extremal rays of $Q_0(\mathcal{T})$. This is an expensive procedure (which is unavoidable, since there is a worst-case exponential number of extremal rays). For this we use the recent state-of-the-art tree traversal method [10], which is tailored to the constraints and pathologies of normal surface theory and is found to be highly effective for larger problems. The tree traversal method works in the larger cone $Q(\mathcal{T})$, but it is a simple matter to insert the two additional linear equations corresponding to $\nu_x = 0$.

  We also note that it is simple to identify the unique closed two-sided normal surface for which $x(S)$ lies on the extremal ray $\mathbf{e}$. Specifically, $x(S)$ is either the smallest integer vector on $\mathbf{e}$ or, if that vector yields a one-sided surface, then its double.
- If we do not reach a conclusive result in step 3, then step 4 requires us to run the Jaco-Tollefson algorithm to test whether any incompressible torus is boundary-parallel. This algorithm is expensive: it requires us to work in a larger normal coordinate system, solve difficult enumeration problems, and perform intricate geometric operations.

  However, it is rare that we should reach this situation, and indeed for all 2979 knots that we consider in Section 5, this scenario never occurs. For some knots (e.g., satellite knots) it cannot be avoided, but there are additional fast methods for avoiding the Jaco-Tollefson algorithm even in these settings, which we describe in the full version of this paper.

THEOREM 6. *Algorithm 5 terminates, and its output is correct.*

PROOF. The algorithm terminates because it does not contain any loops. For correctness, which follows from Theorems 2 and 4, we refer the reader to the full version of this paper. □

5. COMPUTATIONAL RESULTS

Here we describe the results of running the algorithms of Section 4 over significant collections of input knots. These computational results emphasise that the new largeness testing algorithm is both feasible to implement, and fast enough to be practical for non-trivial inputs—both features that distinguish it from many of its peers in algorithmic low-dimensional topology.

The algorithms were implemented in C++ using the software package *Regina* [5, 8]. The code is available from `http://www.maths.uq.edu.au/~bab/code/`, and works with the forthcoming *Regina* version 4.94. Supporting data for the computations described here, including triangulations of the knot complements and the corresponding lists of admissible extreme rays of $Q_0(\mathcal{T})$, can be downloaded from this same location.

All running times reported here are measured on a single 2.93 GHz Intel Core i7 CPU.

5.1 The census of knots up to 12 crossings

Our first data set is the census of all 2977 non-trivial prime knots that can be represented with ≤ 12 crossings. Ideal triangulations of the knot complements were extracted from the *SnapPy* census tables [12], and then further simplified where possible using *Regina*'s greedy heuristics [6] to yield a final set of input triangulations ranging from 2–26 tetrahedra in size.

The algorithms ran successfully over all 2977 triangulations, yielding the following results:

THEOREM 7. *Of the* 2977 *distinct non-trivial prime knots with up to* 12 *crossings,* 1019 *are large and* 1958 *are small.*

A full list of all 1019 large knots can be found in the full version of this paper. Regarding performance:

- The enumeration of extremal rays of $Q_0(\mathcal{T})$ was extremely fast, with a maximum time of 47 seconds, and a median time of just 0.08 seconds. This is a clear illustration of the benefits we obtain from Theorem 2, which allows us to work in the restricted cone $Q_0(\mathcal{T})$ instead of the (typically much larger) cone $Q(\mathcal{T})$.

  The number of extremal rays of $Q_0(\mathcal{T})$ ranged from 0 (for the figure eight knot complement) up to 509 (for one of the 26-tetrahedron triangulations), with a median of 33.

- Testing whether each candidate surface was essential was also extremely fast in most (but not all) cases. For each knot complement, we can sum the times required to process all candidate surfaces: the median sum over all 2977 knots was ~ 3.6 seconds, and all but three of the knots had a processing time of under 12 minutes.

  The remaining three knots, however, were significantly slower to process. One required $\sim$ 3.9 hours, one required $\sim$ 12.2 hours, and one (the knot $12a_{0779}$) was still running after 6 days. However, in a striking illustration of how the algorithms depend strongly upon the underlying triangulations, when the code was run with a different random seed (which affected the simplification heuristics, and hence the triangulations obtained after slicing along surfaces), this worst-case knot $12a_{0779}$ was fully processed in under 4 minutes.

5.2 The dodecahedral knots

We now turn our attention to the dodecahedral knots D_f and D_s as described by Aitchison and Rubinstein [1]. These two knots exhibit remarkable properties [2, 26], and each can be represented with 20 crossings [2]. Running our algorithms over them yields the following results:

THEOREM 8. *The two dodecahedral knots D_f and D_s are both large. In particular, their complements contain closed essential surfaces of genus* 3.

We work with ideal triangulations of the knots D_f and D_s with 46 and 47 tetrahedra respectively, which were kindly provided by Craig Hodgson. These are significantly larger than the knots from the 12-crossing census; indeed, triangulations of this size are typically considered well outside the range of feasibility for normal normal surface theory. Happily our algorithms now prove otherwise:

- This time the enumeration of extremal rays of $Q_0(\mathcal{T})$ was the bottleneck: for D_f and D_s this enumeration took roughly 2.8 and 2.4 days respectively. The number of admissible extremal rays was 72272 and 73609 respectively.

- To test whether candidate surfaces were essential, the knot D_s was completely processed in under 3 minutes; in contrast, D_f required roughly 4.4 hours. Once again, we see that this part of the algorithm depends heavily upon the underlying triangulation: when running with a different random seed, D_f was likewise processed in just a few minutes.

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