Pachner moves, generic complexity, and randomising 3-manifold triangulations

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(joint work with Murray Elder, Jonathan Spreer and Stephan Tillmann)

1. Introduction

We study the computational complexity of decision problems on triangulated 3-manifolds. In this setting there has been encouraging initial progress in recent years, but many important questions remain wide open.

The "simple" problem of 3-sphere recognition and the related problem of unknot recognition are both known to be in NP, by work of Schleimer [18] and earlier work of Hass, Lagarias and Pippenger [9] respectively. In recent announcements by Kuperberg [13] and Hass and Kuperberg [8], these problems are also in co-NP if the generalised Riemann hypothesis holds. It remains a major open question as to whether either problem can be solved in polynomial time.

There are very few hardness results for such problems. A notable example due to Agol, Hass and Thurston is knot genus: if we generalise unknot recognition to computing knot genus, *and* we generalise the ambient space from S^3 to an arbitrary 3-manifold, then the problem becomes NP-complete [1]. The key construction in their result can also be adapted for problems relating to least-area surfaces [1, 7].

2. A hardness result: Taut angle structures

Our first result (in joint work with Spreer) is a hardness result. It relates to taut angle structures on triangulations, as introduced by Lackenby [14]. Taut angle structures are simple and common combinatorial objects; in the right settings they can lead to strict angle structures [11], which are richer objects that in turn can point the way towards building complete hyperbolic structures.

We use the nomenclature of Hodgson et al. [10]: a *taut angle structure* assigns interior angles $\{0, 0, 0, 0, \pi, \pi\}$ to the six edges of each tetrahedron of a triangulation, so that the two π angles are opposite in each tetrahedron, and so that around each edge of the overall triangulation the sum of angles is 2π. Note that this requires the triangulation to be ideal, with torus or Klein bottle vertex links. These structures are slightly more general than the taut structures of Lackenby [14], who also requires consistent coorientations on the 2-faces of the triangulation.

The decision problem that we study is a simple one:

Problem 1 (taut angle structure)**.** *Given an orientable 3-manifold triangulation $\mathcal{T}$ as input, determine whether there exists a taut angle structure on $\mathcal{T}$.*

This decision problem explicitly asks about the geometry of the input triangulation, not the underlying manifold. Our main result is the following:

Theorem 1. taut angle structure *is NP-complete.*

The proof uses a reduction from MONOTONE 1-IN-3 SAT, which was shown by Schaefer to be NP-complete in the 1970s [17]. In MONOTONE 1-IN-3 SAT we have boolean variables $x_1, \ldots, x_t$ and clauses of the form $x_i \vee x_j \vee x_k$, and we must determine whether the t variables can be assigned true/false values so that *precisely* one of the three variables in each clause is true.

For any instance $\mathcal{M}$ of MONOTONE 1-IN-3 SAT, we build a corresponding triangulation that has a taut angle structure if and only if $\mathcal{M}$ is solvable. The triangulation is built by hooking together three types of gadgets: (i) *variable gadgets*, each with two choices of taut angle structure that represent true or false respectively for a single variable x_i of $\mathcal{M}$; (ii) *fork gadgets* that allow us to propagate this choice for x_i to several clauses simultaneously; and (iii) *clause gadgets* that connect three variable gadgets and support an overall taut angle structure if and only if precisely one of the three corresponding variable choices is true. These gadgets have 2, 21 and 4 tetrahedra respectively, and were constructed with significant assistance from the software package *Regina* [2, 5].

This result offers a new framework for proving NP-completeness by building up concrete 3-manifold triangulations, and it is an ongoing project to see how far we can push this framework towards key decision problems such as 0-efficiency testing, 3-sphere recognition, and unknot recognition.

3. Towards an easiness result: 3-sphere recognition

We turn our attention now to 3-sphere recognition. Here we offer a framework (in joint work with Elder and Tillmann), supported by empirical evidence, for solving 3-sphere recognition in polynomial time for *generic* 3-sphere triangulations.

The current state of the art for 3-sphere recognition is outlined in [3] (a cumulation of many results by many authors), and has a running time of $O(7^n \cdot \mathrm{poly}(n))$ in the worst case [6] (Casson describes an $O(3^n \cdot \mathrm{poly}(n))$ solution, but his method is not practical because the 3^n factor is largely unavoidable). Nevertheless, practical implementations are extremely fast in practice: Regina takes just 0.25 milliseconds on average to recognise a 3-sphere triangulation with $n = 10$ tetrahedra.

The key to this speed is *simplification*: Regina will first try to greedily reduce the input triangulation to a small number of tetrahedra using Pachner moves (bistellar flips) [15, 16]; for most 3-sphere triangulations this yields a one-tetrahedron triangulation that can be recognised immediately without running the expensive $O(7^n \cdot \mathrm{poly}(n))$ algorithm at all.

We seek to capture this behaviour using *generic complexity* [12], which allows us to exclude rare pathological inputs from consideration. Let I_n denote all possible inputs of size n. A set of inputs S is called *generic* if $|S \cap I_n|/|I_n| \to 1$ as $n \to \infty$; in other words, the inputs excluded from S become "infinitesimally rare".

We analyse the census of all $31,017,533$ one-vertex 3-sphere triangulations with $n \leq 9$ tetrahedra, and measure paths of Pachner moves between them [4]. Let $p_{n,k}$ denote the probability that a random n-tetrahedron one-vertex 3-sphere triangulation *cannot* be simplified in $\leq k$ such moves. Empirically $p_{n,k}$ falls very fast, at a rate that appears comfortably $O(1/\alpha^{nk})$ for fixed $\alpha > 1$.

Under the right "approximate independence" assumptions, this would allow us to simplify *generic* one-vertex 3-sphere triangulations to a known two-tetrahedron 3-sphere triangulation in polynomial time. The key idea is "aggressive simplification": as our triangulation shrinks we gradually increase the number of allowed moves, allowing us to "jump past" smaller difficult cases as $n \to \infty$, but maintaining an overall polynomial running time of bounded degree in n.

This work ties into the study of random 3-manifold triangulations, where very little is known. Understanding random triangulations—and how to effectively randomise a triangulation of an arbitrary 3-manifold—is an ongoing challenge with significant implications for topological algorithms and complexity.

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