Introducing Regina, the 3-manifold topology software

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Abstract

An overview is presented of *Regina*, a freely available software package for 3-manifold topologists. In addition to working with 3-manifold triangulations, *Regina* includes support for normal surfaces and angle structures. The features of the software are described in detail, followed by examples of research projects in which *Regina* has been used.

1 Introduction

Experimental work in the study of 3-manifold topology has been a historically challenging task. Topological calculations for even simple 3-manifold triangulations are often remarkably difficult to perform by hand. Furthermore, triangulations and 3-manifolds are difficult to represent and manipulate using standard programming languages. Because of this, relatively little software has been available until recently to assist with these calculations.

One prominent exception is *SnapPea* [Weeks 91], under development for over a decade, which provides excellent support for the study of hyperbolic 3-manifolds. More recently a number of other tools have become available, many of which are listed at the computational topology website `http://www.computop.org/`.

Regina is a software package that unites a number of standard 3-manifold topology algorithms and procedures within a friendly user interface, as well as adding previously unavailable features to the current body of experimental tools. Under development since 1999, its growth has to a large extent been guided by its use in a variety of research projects, some of which are noted in Section 3. Algorithms that are implemented include triangulation simplification, normal surface enumeration, angle structure analysis and the calculation of algebraic and combinatorial invariants.

This software is released under the GNU General Public License and is publicly available from `http://regina.sourceforge.net/`. Both the user interface and the underlying programmer's interface are thoroughly documented.

Regina continues to grow and currently enjoys a new release every few months. Special thanks must go to David Letscher who assisted with the early phases of development. Thanks also to Marc Culler, Nathan Dunfield, William Jaco, Richard Rannard, J. Hyam Rubinstein and Jeff Weeks for many fruitful discussions.

In Section 2 we present a detailed description of the capabilities of *Regina*. Section 3 closes with some examples of research projects that have made use of this software.

2 Features

Since its inception, *Regina* has been carefully designed for rigour and extensibility. The software is written primarily in the C++ programming language and runs under GNU/Linux and related operating systems. A list of the more noteworthy features of *Regina* is presented below.

2.1 Triangulations

The primary objects with which a user interacts when running *Regina* are 3-manifold triangulations. As such, a large part of the software is devoted to the creation, analysis and manipulation of triangulations.

2.1.1 Creation

The following methods are supported for creating triangulations.

- Manual construction of triangulations by entering individual tetrahedron face identifications by hand;
- Automatic generation of standard triangulations such as layered solid tori and layered lens spaces [Jaco and Rubinstein 03a, Jaco and Rubinstein 06];
- Automatic construction of Seifert fibred spaces over the 2-sphere with up to three exceptional fibres;
- Reconstruction of triangulations from dehydration strings [Callahan et al. 99];
- Importing triangulations saved from *SnapPea* [Weeks 91].

2.1.2 Analysis

Properties of a triangulation that the software can compute include the following.

- Detailed combinatorial information about the skeleton and boundary components, including vertex links and the shapes formed by the various triangulation faces;
- A variety of homology and homotopy groups;
- The quantum invariants of Turaev and Viro [Turaev and Viro 92];
- 3-sphere recognition, as well as a complete connected sum decomposition for closed orientable triangulations [Jaco and Rubinstein 03a];
- Triangulation attributes relating to the existence of particular types of normal surface, such as 0-efficiency [Jaco and Rubinstein 03a] and the existence of splitting surfaces (described in Section 2.3.3).

Pairs of triangulations can be tested for direct isomorphism, or for whether one triangulation is isomorphic to a subcomplex of another. In addition the software contains a variety of recognition routines for detecting particular well-formed structures within a triangulation. These routines recognise smaller building blocks that often appear within larger triangulations, such as layered solid tori [Jaco and Rubinstein 03a, Jaco and Rubinstein 06] and thin I-bundles [Burton 07]. Furthermore, they can detect complete triangulations belonging to a number of infinite families described in [Burton 03], [Martelli and Petronio 01] and [Matveev 98]. As a result *Regina* can frequently recognise the underlying 3-manifolds for well-structured triangulations that it has not previously encountered.

2.1.3 Manipulation

For the manipulation of a triangulation, the following procedures are available.

- Elementary moves (transformations local to a small number of tetrahedra), such as Pachner moves and other transformations described in [Burton 07], many of which were suggested by Letscher;
- Automated simplification in which the software attempts to use a combination of these elementary moves to reduce the number of tetrahedra as far as possible, though there is no guarantee that the smallest possible number of tetrahedra will be achieved;
- Conversion to a 0-efficient triangulation where possible for closed orientable 3-manifolds [Jaco and Rubinstein 03a];
- Barycentric subdivision and the truncation of ideal vertices (vertices whose links are neither 2-spheres nor discs);
- Conversion of a non-orientable triangulation to an orientable double cover;
- Crushing normal surfaces within a triangulation to a point, as described in Section 2.3.2.

2.2 Census Creation

Regina can form censuses of all 3-manifold triangulations satisfying various sets of constraints. The census algorithm is described in [Burton 07] and contains significant optimisations for censuses of closed minimal $\mathbb{P}^2$-irreducible triangulations. In particular the face pairing graph results of [Burton 04] are incorporated into the algorithm, as are the more standard results relating to low degree edges [Burton 04, Callahan et al. 99, Matveev 98].

Census creation can require significant amounts of computing time (months or years in some cases). As a result, support is provided for splitting this process into pieces and distributing these pieces amongst several machines.

In addition to forming new censuses, *Regina* ships with a number of prepackaged censuses including closed 3-manifolds [Burton 03, Burton 07], cusped hyperbolic 3-manifolds [Callahan et al. 99] and knot and link complements (tabulated by Joe Christy). A census lookup facility for arbitrary triangulations is provided.

2.3 Normal Surfaces

The theory of normal surfaces is a powerful tool for the study of 3-manifolds and for the development of algorithms for their analysis. Normal surfaces were introduced by Kneser [Kneser 29] and further developed by Haken [Haken 61, Haken 62] who used them to construct an algorithm for recognising the unknot. Haken furthermore began the construction of an algorithm for solving the homeomorphism problem for a certain large class of 3-manifolds. Difficulties with the methods of Haken were resolved by Jaco and Oertel and by Hemion [Jaco and Oertel 84, Hemion 92], leading to a finite time algorithm for determining whether two closed irreducible 3-manifolds are homeomorphic in the case in which one of these 3-manifolds contains an embedded two-sided incompressible surface.

Normal surfaces feature in a number of 3-manifold decomposition, homeomorphism and recognition algorithms [Jaco et al. 02, Jaco and Tollefson 95, Rubinstein 95, Rubinstein 97] as well as in algorithms for the simplification of 3-manifold triangulations [Jaco and Rubinstein 03a, Jaco and Rubinstein 03b]. For a more extensive review of normal surface theory, the reader is referred to [Hemion 92].

2.3.1 Creation

Providing a computational tool for the study of normal surfaces was in fact the original motivation behind this software. As such, *Regina* is capable of enumerating all vertex normal surfaces or almost normal surfaces[1] within a triangulation, an operation required by most high-level topological algorithms that utilise normal surface theory.

Regina can perform this vertex enumeration in a variety of coordinate systems. For an n-tetrahedron triangulation this includes the $7n$ standard triangle and quadrilateral coordinates, as well as the smaller set of $3n$ quadrilateral-only coordinates introduced by Tollefson for algorithmic efficiency [Tollefson 98]. The enumeration can be restricted to embedded normal surfaces or can be expanded to include immersed and singular surfaces. Furthermore, elementary support is present for spun normal surfaces, which are non-compact surfaces with infinitely many discs found in ideal triangulations [Tillmann 02].

2.3.2 Analysis

For the analysis of normal surfaces, *Regina* offers the following facilities.

- Viewing normal surfaces in a variety of coordinate systems, including the standard and quadrilateral-only coordinates discussed above as well as the edge weight coordinates introduced by Casson;
- Calculating basic properties of normal surfaces such as Euler characteristic, orientability and one-sidedness;
- Recognising standard surfaces within a triangulation such as splitting surfaces (see Section 2.3.3 below) and vertex and edge links;

[1] Almost normal surfaces are closely related to normal surfaces and are used by Rubinstein in his 3-sphere recognition algorithm [Rubinstein 95, Rubinstein 97].

- Filtering large lists of normal surfaces by various properties such as Euler characteristic, orientability and boundary.

In addition the program can crush a normal surface to a point within a triangulation. Crushing is a powerful tool for the analysis of the role played by a surface within a 3-manifold, and is used in Jaco and Rubinstein's 0-efficiency algorithm [Jaco and Rubinstein 03a].

2.3.3 Splitting Surfaces

Splitting surfaces represent a particular class of normal surfaces whose presence can offer insight into the triangulations containing them. A splitting surface contains precisely one quadrilateral disc within each tetrahedron and no other normal or almost normal discs. These surfaces have a number of interesting combinatorial and topological properties, described in detail in [Burton 03].

As mentioned earlier, *Regina* can detect whether splitting surfaces occur within a triangulation. It also provides support for splitting surface signatures, which are compact text-based representations from which splitting surfaces and their enclosing 3-manifold triangulations can be reconstructed. In addition to performing such reconstructions, the software can form censuses of all possible splitting surface signatures of a given size.

2.4 Angle Structures

Angle structures, studied originally by Casson and then developed by Lackenby and Rivin [Lackenby 00a, Lackenby 00b, Rivin 94, Rivin 03], represent a purely algebraic generalisation of hyperbolic structures. An angle structure on an ideal triangulation is formed by assigning an interior dihedral angle to each edge of every tetrahedron in such a way that a variety of linear equations and inequalities are satisfied.

The formation of angle structures is remarkably similar to the formation of normal surfaces, in which a series of triangle and quadrilateral coordinates are assigned to every tetrahedron with a set of linear equations and inequalities similarly imposed upon them. Thus it has been relatively straightforward to extend the normal surface enumeration code used by *Regina* in such a way that the software can also enumerate vertex angle structures.

Included in the requirements of an angle structure is the condition that each dihedral angle θ satisfies $0 \leq \theta \leq \pi$. In addition to the enumeration of vertex angle structures, *Regina* can identify whether a triangulation supports any strict angle structures (for which each dihedral angle θ satisfies $0 < \theta < \pi$) or any taut angle structures (for which each dihedral angle is precisely 0 or π).

2.5 Scripting

Regina offers the ability to write and run arbitrary scripts in the *Python* scripting language. These scripts are essentially high-level programs with immediate access to the mathematical core of *Regina*, and are ideal for performing repetitive tasks over large sets of data. Such tasks might include performing a sequence of tests upon all triangulations in a census, or testing a prototype for a new algorithm. *Regina* data files can contain embedded scripts, and different files can share code through the use of external libraries of routines.

2.6 Interfaces and Documentation

The usual method of running *Regina* provides a full graphical interface that a user can easily understand and use. Alternatively, for those requiring immediate access to the mathematical core of the software, an interactive command-line interface is offered from which users can control the program using the *Python* scripting language described above. A variety of specialised utility programs are also available.

Significant effort has been spent on documentation for the software. A full reference manual is available for end users to assist them in working with *Regina*. This reference manual can be read online at `http://regina.sourceforge.net/docs/`. For users writing *Python* scripts or for programmers seeking to modify or extend the software, the routines offered by the underlying mathematical core are also fully documented.

2.7 Data Files

The data files used for saving triangulations and other information adhere to a well-organised hierarchical structure. This structure not only allows multiple triangulations, normal surface lists and other topological structures to be stored together in an organised fashion but it also supports the storing of miscellaneous data such as text notes and *Python* scripts. The file format is well documented in the reference manual and uses compressed XML[2], allowing for the simple transfer of native *Regina* data to and from other programs.

3 Applications

We close with some examples of research projects in which *Regina* has been used with success.

- In [Burton 07] a census is presented of all closed non-orientable minimal $\mathbb{P}^2$-irreducible triangulations formed from ≤ 7 tetrahedra. Computational support from *Regina* was required not only for the formation of the census but also for the detailed combinatorial analysis of the resulting triangulations. A similar census of orientable triangulations appears in [Burton 03], again relying upon *Regina* for much computational support.
- Various constraints upon the structures of minimal triangulations are proven in [Burton 04]. For this research *Regina* was used to obtain and process data that originally motivated the results, as well as to measure the subsequent improvements to the census algorithm.
- Research into the existence of taut angle structures on ideal triangulations is described in [Burton et al. 03]. Here *Regina* was used to process large bodies of census data to locate and subsequently analyse triangulations that do not support taut structures.
- For the studies of 0-efficiency and 1-efficiency described in [Jaco and Rubinstein 03a] and [Jaco and Rubinstein 03b], *Regina* has assisted with the construction and analysis of pathological triangulations.

Acknowledgements

The author would like to acknowledge the support of the American Institute of Mathematics, the Australian Research Council and the Grayce B. Kerr Chair at Oklahoma State University.

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[2] XML is the *Extensible Markup Language*, an open and widely-supported text-based data format.

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