ON THE ROOTS OF $z^n + te^{i\alpha}z^{n-1} + te^{-i\alpha}z + 1 = 0$

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Abstract. The study of Bethe-type equations in Mathematical physics leads to the study of the roots of polynomials with a specific structure. Most of the roots of these polynomials lie on the unit circle. However, as a parameter increases each such polynomial acquires precisely one pair of roots conjugate with respect to the unit circle.

In an earlier paper, the first author considered Baxter's formulae for the transfer matrix spectrum of the superintegrable Z_N-chiral Potts model for skewed boundary conditions. In deriving the m_P-structure of the low-lying energy levels in the different sectors of the quantum chain with skewed boundary conditions at high- and low-temperatures, numerical solutions of the Bethe-type equations were obtained.

For $m_P = 2$ the numerical procedures indicated that the solutions of the equations possessed a very simple structure.

In this paper, these properties are derived analytically.

This problem originally occurs in the form

$$a_n x^n + \zeta(a_{n-1}x^{n-1} + a_{n-1}^* x) + a_n^* = 0 \ ,$$

where $n \geq 3$ and ζ is a real non-negative parameter.

However, setting $a_n = ae^{i\phi}$, $a_{n-1} = be^{i\psi}$ and $x = ze^{-2i\theta}$, this equation becomes

$$e^{-in\theta}\left(ae^{i(\phi-n\theta)}z^n + \zeta b(e^{i(\psi-(n-2)\theta)}z^{n-1} + e^{-(\psi-(n-2)\theta)}z) + ae^{-i(\phi-n\theta)}\right) = 0 \ .$$

Choosing $\theta = \phi/n$, $\alpha = \psi - (n-2)\theta$ and $t = \zeta b/a$, reduces the equation to the form

$$f(z) \ \equiv \ z^n + t\left(e^{i\alpha}z^{n-1} + e^{-i\alpha}z\right) + 1 = 0 \ ,$$

where t is still a real non-negative parameter.

Note first that

$$\begin{aligned} f\left(\frac{1}{z^*}\right) &= (z^*)^{-n} + te^{i\alpha}(z^*)^{1-n} + te^{-i\alpha}(z^*)^{-1} + 1 \\ &= (z^*)^{-n}\left(1 + te^{i\alpha}z^* + te^{-i\alpha}(z^*)^{n-1} + (z^*)^n\right) \\ &= (z^*)^{-n}\left(f(z)\right)^* \end{aligned}$$

Therefore, if z is a root of the polynomial, so is $1/z^*$. This means that the roots are either of the form $e^{i\theta}$ or occur in pairs $\sigma e^{i\theta}$, $(1/\sigma)e^{i\theta}$.

When $t = 0$, the equation simplifies to $z^n + 1 = 0$, so that all the roots are of the first type, and are evenly spaced around the unit circle in the complex plane. Since the roots are continuous functions of the parameter t, the roots will remain on the unit circle as t increases until two of the roots coalesce. When this coalescence occurs, f and f' vanish simultaneously on the unit circle. For smaller values of t, we note that the zeros of f' lie in the convex hull of the zeros of f and hence inside or on the unit circle. However, a zero of f' cannot lie on the unit circle without coinciding with a root of f. From this we conclude that no coalescence can occur as long as the zeros of f' lie within the unit circle.

Applying Rouché's Theorem to

$$nz^{n-1} + te^{i\alpha}\left((n-1)z^{n-2} + e^{-2i\alpha}\right)$$

on the unit circle, we see that

$$\left|te^{i\alpha}\left((n-1)e^{i(n-2)\theta} + e^{-2i\alpha}\right)\right| \le t((n-1)+1) = nt$$

so that no coalescence can occur for $t < 1$.

Consideration of the equation

$$z^n - t(z^{n-1} + z) + 1 = 0$$

shows that this bound is the best possible.

Suppose that $z = e^{2i\theta}$, $0 \le \theta < \pi$, is a root of the polynomial. Then

$$e^{in\theta}\left(e^{in\theta} + te^{i((n-2)\theta+\alpha)} + te^{-i(n-2)\theta+\alpha)} + e^{-in\theta}\right) = 0$$

$$\cos(n\theta) + t\cos((n-2)\theta + \alpha) = 0$$

If we have a multiple root, then

$$-n\sin(n\theta) - (n-2)t\sin((n-2)\theta + \alpha) = 0$$

also. Combining these expressions, we see that for coalescence

$$\begin{aligned} t\cos((n-2)\theta+\alpha) &= -\cos(n\theta) \\ t^2\cos^2((n-2)\theta+\alpha) &= \cos^2(n\theta) \\ t\sin((n-2)\theta+\alpha) &= -\frac{n}{n-2}\sin(n\theta) \\ t^2\sin^2((n-2)\theta+\alpha) &= \frac{n^2}{(n-2)^2}\sin^2(n\theta) \\ t^2 &= \cos^2(n\theta) + \frac{n^2}{(n-2)^2}\sin^2(n\theta) \\ 1 \le t^2 &\le \frac{n^2}{(n-2)^2} \\ 1 \le t &\le \frac{n}{n-2} \end{aligned}$$

Again, these bounds are sharp.

If we consider the case

$$\cos(n\theta) = 0 \; ; \; \cos((n-2)\theta + \alpha) = 0$$

the second equation gives

$$\pm n \pm (n-2)t = 0 \; .$$

Since $t \geq 0$, we must have $\sin(n\theta)$ and $\sin((n-2)\theta + \alpha)$ of opposite sign and $t = n/(n-2)$. This case arises when

$$\begin{aligned} n\theta &= \left(k + \frac{1}{2}\right)\pi \\ (n-2)\theta + \alpha &= \left(l + \frac{1}{2}\right)\pi \; ; \; (k-l) \text{ odd} \\ 2\theta - \alpha &= (k-l)\pi \end{aligned}$$

If we specify $-\pi \leq \alpha < \pi$, this equation has a unique solution for θ in the range $0 \leq \theta < \pi$ given by

$$\theta = \frac{1}{2}(\alpha + \pi)$$

corresponding to $k - l = 1$. There are n such solution pairs given by

$$\begin{aligned} \theta &= \frac{2k+1}{2n}\pi \; ; \; k = 0 \ldots n-1 \\ \alpha &= \left(\frac{2k+1}{n} - 1\right)\pi \end{aligned}$$

Similarly, the case

$$\sin(n\theta) = 0 \; ; \; \sin((n-2)\theta + \alpha) = 0$$

gives $t = 1$ for the n pairs of values

$$\begin{aligned} \theta &= \frac{k}{n}\pi \; ; \; k = 0 \ldots n-1 \\ \alpha &= \left(\frac{2k}{n} - 1\right)\pi \end{aligned}$$

where again

$$\theta = \frac{1}{2}(\alpha + \pi) \; .$$

For the other values of α we can eliminate t from the equations, giving the transcendental equation

$$\cos\alpha((n-1)\sin 2\theta + \sin 2(n-1)\theta) = \sin\alpha((n-1)\cos 2\theta - \cos 2(n-1)\theta)$$

for θ. Using this equation in turn to eliminate α gives

$$\begin{aligned} t &= \sqrt{1 + 4(n-1)\sin^2 n\theta/(n-2)^2} \qquad (\geq 1) \\ &= \frac{n}{n-2}\sqrt{1 - 4(n-1)\cos^2 n\theta/n^2} \qquad \left(\leq \frac{n}{n-2}\right) \end{aligned}$$

Approximate solutions of these equations can be found by perturbing about the solutions found above.

If we set

$$\alpha = \left(\frac{2k+1}{n} - 1\right)\pi + \beta$$
$$\theta = \frac{2k+1}{2n}\pi + \phi$$
$$t = \frac{n}{n-2} - \tau$$

where β, ϕ and τ are *small*, we obtain

$$\beta \sim 2(n-1)(n-2)\phi^3$$
$$\tau \sim \frac{2n(n-1)}{n-2}\phi^2 \;\sim\; c\beta^{2/3}$$

independent of the value of k.

If we set

$$\alpha = \left(\frac{2k}{n} - 1\right)\pi + \beta$$
$$\theta = \frac{k}{n}\pi + \phi$$
$$t = 1 + \tau$$

we obtain

$$\beta \sim \frac{4(n-1)}{n-2}\phi$$
$$\tau \sim \frac{2n^2(n-1)}{(n-2)^2}\phi^2$$

again independent of the value of k.

For large t, we can again consider Rouché's Theorem, this time comparing $F(z) = t(e^{i\alpha}z^{n-1} + e^{-i\alpha}z)$ and $g(z) = z^n + 1$.

On the circle $|z| = \rho$, $|F| \geq t\rho|\rho^{n-2} - 1|$ and $|g| \leq \rho^n + 1$. Hence, for $\rho < 1$, we see that our original polynomial f has precisely one root inside the circle $|z| = \rho$ provided

$$t\rho(1 - \rho^{n-2}) > 1 + \rho^n$$

and for $\rho > 1$ has $n-1$ roots inside the circle $|z| = \rho$ provided

$$t\rho(\rho^{n-2} - 1) > 1 + \rho^n$$

Hence, in the limit as $t \to \infty$, the polynomial has $n-2$ roots on the unit circle, and a single pair of roots of the form $\sigma e^{i\theta}$, $(1/\sigma)e^{i\theta}$, where $\sigma \to 0$ as $t \to \infty$.

Since the structural form of the roots can only change at a coalescence, this behaviour holds for $t > n/(n-2)$.

An expression for the root lying inside the unit circle can be obtained using the Lagrange expansion.

Rewriting the equation as

$$\begin{aligned} te^{-i\alpha}z(1 + e^{2i\alpha}z^{n-2}) &= -(1 + z^n) \\ z &= \left(-\frac{e^{i\alpha}}{t}\right)\frac{1 + z^n}{1 + e^{2i\alpha}z^{n-2}} \end{aligned}$$

and noting that as $t \to \infty$, $z \to 0$, we obtain

$$z = \sum_{r=1}^{\infty} \frac{1}{r!}\left(-\frac{e^{i\alpha}}{t}\right)^r D^{r-1}\left(\frac{1 + z^n}{1 + e^{2i\alpha}z^{n-2}}\right)^r \Bigg|_{z=0} .$$

This expansion converges for t greater than its bifurcation value.

With $s = -e^{i\alpha}/t$, we have

$$z \sim s - e^{2i\alpha}s^{n-1} + s^{n+1}$$

as $t \to \infty$.