Not quite the pantograph equation

We consider the functional differential equation

$$f'(t) = af(at)\ ;\ f(0) = 1 \tag{1}$$

where $0 < a < 1$. The pantograph equation has an extra $f(t)$ term on the right hand side, and this term dominates the asymptotic behaviour of the solutions of that equation.

The equation (1) has the power series solution

$$f(t) = \sum_{n=0}^{\infty} a^{n(n+1)/2} \frac{t^n}{n!}$$

which is an entire function for $|a| \leq 1$, and is monotonic increasing on $[0, \infty)$ for $0 < a < 1$.

When $a = 0$ the solution reduces to $f \equiv 1$ and when $a = 1$, $f = e^t$.

(Closed forms for the solution are also available when $a = \exp(p\pi i/q)$; in particular, when $a = -1$,

$$f(t) = \cos(t) - \sin(t)$$

and when $a = i$

$$f(t) = \phi(t) + \phi'(t) + i(\phi''(t) - \phi'''(t))$$

where $\phi(t) = \cosh(t/\sqrt{2})\cos(t/\sqrt{2})$.)

Of interest is the asymptotic behaviour of the solution as $t \to \infty$ for $0 < a < 1$.

We can write the equation in the integral form

$$f(t) = 1 + \int_0^{at} f(s)\, ds \tag{2}$$

Since $f(t) < e^t$ for $t > 0$, we have successively

$$\begin{aligned} f(t) &< e^{at} \\ f(t) &< a^{-1}e^{a^2 t} - (a^{-1} - 1) \\ f(t) &< a^{-3}e^{a^3 t} - (a^{-3} - 1) + (1 - a)t \end{aligned}$$

and in general

$$f(t) < \ a^{-n(n-1)/2}e^{a^n t} + P_{n-2}(t)$$

where P is a polynomial.

Since $0 < a < 1$, $a^n \to 0$ as $n \to \infty$, so that for any $\epsilon > 0$, $e^{-\epsilon t} f(t) \to 0$ as $t \to \infty$.

On the other hand, f has better than polynomial growth as $t \to \infty$, so that we expect that

$$f(t) \sim t^\alpha e^{\beta t^r}\ ,\ 0 < r < 1$$

as $t \to \infty$.

(When $a = \frac{2}{3}$, I suspect $\alpha = -\frac{\sqrt{3}}{2}$, $\beta = 2$, and $r = \frac{1}{2}$.)