1(a) Determine the characteristic of the operator

$$x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y}$$

which passes through the point $(1,1)$.

Ans The characteristic equations are

$$\begin{aligned} \frac{dx}{dt} &= x \ ; \ x(0) = 1 \\ x &= e^t \\ \frac{dy}{dt} &= 2y \ ; y(0) = 1 \\ y &= e^{2t} \ = \ x^2 \ ; \ x > 0 \ . \end{aligned}$$

■

(b) If

$$x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = 3u \ ,$$

and $u(1,1) = 2$, determine $u(2,4)$.

Ans

$$\begin{aligned} \frac{du}{dt} &= 3u \ ; u(0) = 2 \\ u &= 2e^{3t} \\ \text{When} \qquad x &= e^t \ = \ 2 \ ; \ u \ = \ 2(2)^3 \ = \ 16 \end{aligned}$$

■

2. Determine the solution of the quasilinear initial value problem

$$x\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = y \ ,$$
$$u(1,y) = 2y \ .$$

Ans Parametrising the initial data, we have

$$x = 1 \ ; \ y = s \ ; u = 2s$$

The characteristic equations are

$$\begin{aligned}
\frac{dx}{dt} &= x \ ; \ x(0) = 1 \\
x &= e^t \\
\frac{dy}{dt} &= u \ ; y(0) = s \\
\frac{du}{dt} &= y \ ; u(0) = 2s \\
\frac{d}{dt}(y+u) &= y + u \ ; (y+u)(0) = 3s \\
y + u &= 3se^t \ = 3sx \\
\frac{d}{dt}(y-u) &= \ -(y-u) \ ; \ (y-u)(0) = -s \\
y - u &= \ -se^{-t} = -\frac{s}{x} \\
\frac{y+u}{3x} &= s \ = x(u-y) \\
u + y &= 3x^2u - 3x^2y \\
(3x^2 - 1)u &= (3x^2+1)y \\
u &= \frac{3x^2+1}{3x^2-1}y \ ; \ x > \frac{1}{\sqrt{3}}
\end{aligned}$$

The base characteristics are given by

$$y = \frac{s}{2}(3e^t - e^{-t}) = \frac{s}{2}\left(3x - \frac{1}{x}\right)$$

which represent a family of hyperbolae all of which pass through

$$x = \frac{1}{\sqrt{3}} \ , \ y = 0 \ .$$

Therefore the solution is defined only for $x > 1/\sqrt{3}$. ■

3. Solve the semi-infinite wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \ ; \ t > 0 \ ; x > 0$$

subject to the initial conditions

$$u(x,0) = e^{-x} \ ; \frac{\partial u}{\partial t}(x,0) = 0$$

and the boundary condition

$$u(0,t) = e^{-t} \ .$$

Ans For $x > t$ we have d'Alembert's solution

$$u(x,t) = \frac{1}{2}e^{-x-t} + \frac{1}{2}e^{-x+t} = e^{-x}\cosh t$$

For $0 < x < t$, we have

$$\begin{aligned}
u(x,t) &= \phi(t-x) + \frac{1}{2}e^{-x-t} \\
u(0,t) &= \phi(t) + \frac{1}{2}e^{-t} = e^{-t} \\
\phi(t) &= \frac{1}{2}e^{-t} \\
u(x,t) &= \frac{1}{2}e^{x-t} + \frac{1}{2}e^{-x-t} = e^{-t}\cosh x
\end{aligned}$$

■