MATH3403
FINAL EXAMINATION 2002
SOLUTIONS

1. Use Dirichlet's principle to obtain an approximate solution of the boundary value problem

$$u'' = u \ , \quad u(0) = 0 \ , \ u(1) = 1 \ ,$$

starting with the form

$$w = x + a\sin(\pi x) \ .$$

Ans

$$\begin{aligned}
\langle u|v\rangle &= \int_0^1 (u'v' + uv)\,dx \\
w_0 &= x \ ; \ v = \sin(\pi x) \\
\langle w_0|v\rangle &= \int_0^1 (\pi\cos(\pi x) + x\sin(\pi x))\,dx \\
&= \sin(\pi x) - \frac{x}{\pi}\cos(\pi x)\Big|_0^1 + \frac{1}{\pi}\int_0^1 \cos(\pi x)\,dx \\
&= \frac{1}{\pi} + \frac{1}{\pi^2}\sin(\pi x)\Big|_0^1 = \frac{1}{\pi} \\
\langle v|v\rangle &= \int_0^1 \left(\pi^2\cos^2(\pi x) + \sin^2(\pi x)\right)\,dx \\
&= \frac{1+\pi^2}{2} \\
\frac{1}{\pi} &+ \frac{1+\pi^2}{2}a = 0 \\
a &= -\frac{2}{\pi + \pi^3} \\
w &= x - \frac{2\sin(\pi x)}{\pi + \pi^3}
\end{aligned}$$

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2. By considering $u = F(\eta)$, where $\eta = x/\sqrt{t}$, find the solution of the heat equation

$$u_{xx} = u_t \ ; \ x > 0 \ , \ t > 0$$

for which $u(x, 0) = 1$ and $u(0, t) = 0$, in the form of an integral.

Ans

$$u_x = \frac{1}{\sqrt{t}} F'(\eta)$$
$$u_{xx} = \frac{1}{t} F''(\eta)$$
$$u_t = -\frac{1}{2} \frac{x}{t^{3/2}} F'(\eta)$$
$$F'' = -\frac{\eta}{2} F'$$
$$F' = c e^{-\eta^2/4}$$
$$F = d + c \int_0^{\eta} e^{-s^2/4}\, ds$$
$$x = 0 \to \eta = 0 \ ; \ F(0) = 0 \ ; d = 0$$
$$t = 0 \to \eta = \infty \ ; F(\infty) = 1$$
$$c \int_0^{\infty} e^{-s^2/4}\, ds = 1$$
$$c = \frac{1}{\sqrt{\pi}}$$
$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{t}} e^{-s^2/4}\, ds$$

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3. Use an integral transform to determine the solution of the following problem.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad ; \quad 0 < x < \infty \ , \ 0 < t < \infty \ ;$$
$$u(x,0) = 0 \ , \ u_t(x,0) = 0 \ , \ u_x(0,t) = 1 \ .$$

Ans Using a Fourier Cosine Transform:

$$U(\omega,t) = \int_0^{\infty} \cos(\omega x) u(x,t)\, dx$$
$$U_{tt} = -\omega^2 U - 1$$
$$U(\omega,0) = 0 \ ; \ U_t(\omega,0) = 0$$
$$U(\omega,t) = -\frac{1}{\omega^2}(1 - \cos(\omega t))$$
$$u(x,t) = \begin{cases} x - t & 0 \le x \le t \\ 0 & t > x \end{cases}$$

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