1. (**10 marks**) Find the solution of the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{4}\frac{\partial^2 u}{\partial t^2} = x^2 t^2$$

in the region $x \geq 0$, $t \geq 0$, subject to the initial conditions

$$u(x,0) = 0 \ ; \ u_t(x,0) = 0$$

and the boundary condition

$$u(0,t) = 0 \ .$$

Ans: $c = 2$.
Using the integral formula, we have the particular solution

$$\begin{aligned}
u(x,t) &= -\frac{c}{2}\int_0^t \tau^2\, d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} \sigma^2\, d\sigma \\
&= -\int_0^t \tau^2\, d\tau \left[\frac{1}{3}(x+2(t-\tau))^3 - \frac{1}{3}(x-2(t-\tau))^3\right] \\
&= -\int_0^t \tau^2\, d\tau \left[4x^2(t-\tau) + \frac{16}{3}(t-\tau)^3\right] \\
&= -4x^2\int_0^t (t\tau^2 - \tau^3)\, d\tau - \frac{16}{3}\int_0^t (t^3\tau^2 - 3t^2\tau^3 + 3t\tau^4 - \tau^5)\, d\tau \\
&= -4x^2t^4\left(\frac{1}{3} - \frac{1}{4}\right)) - \frac{16}{3}t^6\left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6}\right) \\
&= -\frac{1}{3}x^2t^4 - \frac{4}{45}t^6
\end{aligned}$$

This solution satisfies the homogeneous initial conditions, and is the required solution for $x > 2t$.

When $x = 0$, the particular solution is $-\frac{4}{45}t^6$, therefore to satisfy the boundary condition we add $\frac{4}{45}(t - \frac{1}{2}x)^6$ in the region $0 < x < 2t$.

$$u(x,t) = \begin{cases} -\frac{1}{3}x^2t^4 - \frac{4}{45}t^6 & x > 2t \\ \frac{4}{45}((t-\frac{1}{2}x)^6 - t^6) - \frac{1}{3}x^2t^4 & 0 < x < 2t \end{cases}$$

2. (**5 marks**) Derive the adjoint operator $\mathcal{L}^*$ to the operator

$$\mathcal{L}(y) = p_2(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_0(x)y$$

on the interval $a \leq x \leq b$.

Under what condition is $\mathcal{L}^* = \mathcal{L}$?

$$\begin{aligned}
\int_a^b v\mathcal{L}(y)\,dx &= \int_a^b [(vp_2)y'' + (vp_1)y' + (vp_0)y]\,dy \\
&= [(vp_2)y' + (vp_1)y]_a^b \\
&\quad + \int_a^b [-(vp_2)'y' - (vp_1)'y + (vp_0)y]\,dx \\
&= [(vp_2)y' + (vp_1)y - (vp_2)'y]_a^b \\
&\quad + \int_a^b [(vp_2)''y - (vp_1)'y + (vp_0)y]\,dx \\
&= [(vp_2)y' + (vp_1)y - (vp_2)'y]_a^b \\
&\quad + \int_a^b y[p_2v'' + (2p_2' - p_1)v' + (p_2'' - p_1' + p_0)v]\,dx
\end{aligned}$$

$$\mathcal{L}^*(v) = p_2v'' + (2p_2' - p_1)v' + (p_2'' - p_1' + p_0)v$$

If $\mathcal{L} = \mathcal{L}^*$,

$$\begin{aligned}
p_2 &= p_2 \\
2p_2' - p_1 &= p_1 \;;\; p_1 = p_2' \\
p_2'' - p_1' + p_0 &= p_0 \;;\; p_1' = p_2''
\end{aligned}$$

which is satisfied if $p_1 = p_2'$.

3. (**15 marks**) Find the solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, subject to the boundary conditions

$$\begin{aligned} u(x,0) &= 0 \; ; \; u(x,1) &= 0 \\ u(0,y) &= 0 \; ; \; u(1,y) &= y(1-y) \end{aligned}$$

Ans Consider the separated solution $X(x)Y(y)$ where $Y(0) = Y(1) = 0$.
The eigenfunctions $Y(y)$ satisfy

$$\begin{gathered} Y'' + \omega^2 Y = 0 \; ; \; Y(0) = Y(1) = 0 \\ Y = A\sin(\omega x) + B\cos(\omega x) \\ Y(0) = B = 0 \\ Y(1) = A\sin(\omega) = 0 \end{gathered}$$

so that for non-trivial solution

$$\begin{gathered} \omega = \omega_n = n\pi \; ; \; n = 1, 2, \ldots \\ Y_n(y) = \sin(n\pi y) \end{gathered}$$

The corresponding amplitudes $X_n(x)$ satisfy

$$\begin{gathered} X_n'' - (n\pi)^2 X_n = 0 \\ X_n = a_n \sinh(n\pi x) + b_n \cosh(n\pi x) \end{gathered}$$

If we look for a solution in the form

$$u(x,y) = \sum_{n=1}^{\infty} (a_n \sinh(n\pi x) + b_n \cosh(n\pi x)) \sin(n\pi x)$$

then

$$u(0,y) = \sum_{n=1}^{\infty} b_n \sin(n\pi y) = 0$$
$$b_n = 0 \ \forall \ n$$
$$u(1,y) = \sum_{n=1}^{\infty} a_n \sinh(n\pi) \sin(n\pi y) = y(1-y)$$
$$a_n \sinh(n\pi) \int_0^1 \sin^2(n\pi y)\, dy = \int_0^1 y(1-y) \sin(n\pi y)\, dy$$
$$\frac{1}{2}\sinh(n\pi) a_n = -\frac{1}{n\pi} y(1-y)\cos(n\pi y)\Big|_0^1 + \frac{1}{n\pi}\int_0^1 (1-2y)\cos(n\pi y)\, dy$$
$$= \frac{1}{n^2\pi^2}(1-2y)\sin(n\pi y)\Big|_0^1 + \frac{2}{n^2\pi^2}\int_0^1 \sin(n\pi y)\, dy$$
$$= \frac{2}{n^3\pi^3}(\cos(0) - \cos(n\pi)) = \frac{2}{n^3\pi^3}(1-(-1)^n)$$
$$a_n = \begin{cases} 0 & n = 2m \\ \frac{8}{\sinh((2m+1)\pi)} \frac{1}{\pi^3(2m+1)^3} & n = 2m+1 \end{cases}$$
$$u(x,y) = \frac{8}{\pi^3} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^3} \frac{\sinh((2m+1)\pi x)}{\sinh((2m+1)\pi)} \sin((2m+1)\pi y)$$