1. Determine the eigenvalues and eigenfunctions for the equation

$$y'' + \lambda y = 0 \ ; \ y(0) = y(1) = 0 \ ,$$

and expand the function $f(x) = x - x^2$ in terms of these eigenfunctions.

Ans. Since the boundary conditions are Dirichlet, the eigenvalues are positive.
Let $\lambda = \omega^2$. The general solution of $y'' + \omega^2 y = 0$ is

$$y = A\cos\omega x + B\sin\omega x \ .$$

If $y(0) = 0$, then $A = 0$.
If $y(1) = 0$,

$$\begin{aligned} B\sin\omega &= 0 \\ \sin\omega &= 0 \quad \text{if } B \neq 0 \\ \omega &= n\pi \quad n = 1, 2, \ldots \\ \lambda &= n^2\pi^2 \\ y &= \sin(n\pi x) \end{aligned}$$

If $x - x^2 = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$, then

$$\begin{aligned} a_n &= \int_0^1 (x - x^2)\sin(n\pi x)\,dx \Big/ \int_0^1 \sin^2(n\pi x)\,dx \\ &= 2\left[-(x - x^2)\frac{1}{n\pi}\cos(n\pi x)\Big|_0^1 + \frac{1}{n\pi}\int_0^1 (1 - 2x)\cos(n\pi x)\,dx \right] \\ &= \frac{2}{n^2\pi^2}(1 - 2x)\sin(n\pi x)\Big|_0^1 + \frac{4}{n^2\pi^2}\int_0^1 \sin(n\pi x)\,dx \\ &= -\frac{4}{n^3\pi^3}\cos(n\pi x)\Big|_0^1 \\ &= \frac{4(1 - (-1)^n)}{n^3\pi^3} \end{aligned}$$

$$x - x^2 = \frac{8}{\pi^3}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3}\sin((2k+1)\pi x)$$

2. Determine the solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 1$$

in the square $0 \le x \le 1$, $0 \le y \le 1$, subject to the boundary conditions

$$u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0 \ .$$

(**Hint:** Find a particular solution as a function of x alone, and use separation of variables to determine the complementary function.)

Ans. If $U(x)$ is a particular solution which satisfies the boundary conditions at $x = 0$ and $x = 1$,

$$\begin{aligned} U'' &= 1 \\ U &= \frac{1}{2}x^2 + ax + b \\ U(0) &= b = 0 \\ U(1) &= \frac{1}{2} + a = 0 \ ; \ a = -\frac{1}{2} \end{aligned}$$

Let $u(x,t) = v(x,t) + U(x)$. Then

$$\begin{aligned} v_{xx} + v_{yy} &= 0 \\ v(x,0) = v(x,1) &= \frac{1}{2}(x - x^2) \\ v(0,y) = v(1,y) &= 0 \end{aligned}$$

If $X(x).Y(y)$ satisfies

$$v_{xx} + v_{yy} = 0 \ ; \ v(0,y) = v(1,y) = 0$$

then

$$X'' + \lambda X = 0 \ ; \ X(0) = X(1) = 0$$

so from question 1, $\lambda = n^2\pi^2$ and $X_n = \sin(n\pi x)$.

The functions $Y_n(y)$ satisfy

$$Y'' - n^2\pi^2 Y = 0 \ ; \ Y_n = A_n e^{n\pi y} + B_n e^{-n\pi y}$$

Set

$$v(x,y) = \sum_n^\infty \sin(n\pi x)(A_n e^{n\pi y} + B_n e^{-n\pi y})$$

$$v(x,0) = \frac{1}{2}(x - x^2) = \sum_{n=1}^{\infty}(A_n + B_n)\sin(n\pi x)$$

$$A_n + B_n = \frac{1}{2}a_n$$

$$v(x,1) = \frac{1}{2}(x - x^2) = \sum_{n=1}^{\infty}(A_n e^{n\pi} + B_n e^{-n\pi})\sin(n\pi x)$$

$$e^{n\pi}A_n + e^{-n\pi}B_n = \frac{1}{2}a_n$$

$$A_n = \frac{e^{-n\pi} - 1}{e^{-n\pi} - e^{n\pi}}\frac{1}{2}a_n = \frac{1}{2}\frac{1}{1 + e^{n\pi}}a_n$$

$$B_n = \frac{1 - e^{n\pi}}{e^{-n\pi} - e^{n\pi}}\frac{1}{2}a_n = \frac{1}{2}\frac{e^{n\pi}}{1 + e^{n\pi}}a_n$$

$$v(x,t) = \frac{4}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\frac{e^{(2k+1)\pi y} + e^{(2k+1)\pi(1-y)}}{1 + e^{(2k+1)\pi y}}\sin((2k+1)\pi x)$$

$$u(x,t) = v(x,t) + \frac{1}{2}(x^2 - x)$$

3. Suppose that f is a harmonic function in the circle, centre $\boldsymbol{c}$ and radius R in $\mathbb{R}^2$. Show that for every $0 < r < R$,

$$f(\boldsymbol{c}) = \frac{1}{2\pi r}\oint_{|\boldsymbol{x}-\boldsymbol{c}|=r} f(\boldsymbol{x})\,ds\ .$$

You may assume that if f is harmonic in $\mathcal{R}$, then

$$\oint_{\partial\mathcal{R}}\frac{\partial f}{\partial n}\,ds = 0\ .$$

Ans. See Week 6, page 4.