1. Solve the following first order partial differential equation, indicating the region for which the solution is defined.

$$xu_x + yu_y = u \quad u(x,1) = \sqrt{x}\ , \quad x > 0\ .$$

Ans.
The initial data is $x(0) = s$, $y(0) = 1$, $u(0) = \sqrt{s}$.
The characteristic equations are

$$\begin{aligned}
\frac{dx}{dt} &= x\ ;\ x = se^t\ (> 0) \\
\frac{dy}{dt} &= y\ ;\ y = e^t\ (> 0) \\
s &= \frac{x}{y} \\
\frac{du}{dt} &= u\ ;\ u = \sqrt{s}e^t = \sqrt{\frac{x}{y}}y = \sqrt{xy}
\end{aligned}$$

The solution is defined for $x > 0$, $y > 0$.

2. Reduce the equation

$$u_{xx} - 5u_{xy} + 6u_{yy} = 0$$

to the appropriate canonical form, and hence find the solution of the equation for which

$$u(0,y) = e^y\ ;\ u_x(0,y) = 0\ .$$

Ans.

$$X^2 - 5X + 6 = (X-2)(X-3)$$

therefore the characteristic variables satisfy

$$\begin{gathered}
\xi_x - 2\xi_y = 0 \\
\frac{dx}{dt} = 1\ ,\ \frac{dy}{dt} = -2\ ,\ \frac{dy}{dx} = -2 \\
y = -2x + c\ ;\ \xi = y + 2x
\end{gathered}$$

$$\begin{gathered}
\eta_x - 3\eta_y = 0 \\
\frac{dx}{dt} = 1\ ,\ \frac{dy}{dt} = -3\ ,\ \frac{dy}{dx} = -3 \\
y = -3x + c\ ;\ \eta = y + 3x
\end{gathered}$$

In terms of the characteristic variables, the equation reduces to

$$u_{\xi\eta} = 0$$

for which the general solution is

$$\begin{aligned}
u(x,y) &= f(\xi)+g(\eta) \\
&= f(y+2x)+g(y+3x) \\
u_x(x,y) &= 2f'(y+2x)+3g'(y+3x) \\
u(0,y) &= f(y)+g(y)=e^y \\
u_x(0,y) &= 2f'(y)+3g'(y)=0 \\
2f(y)+3g(y) &= 0 \\
2f(y)+2g(y) &= 2e^y \\
g(y) &= -2e^y \\
f(y) &= 3e^y \\
u(x,y) &= 3e^{y+2x}-2e^{y+3x}
\end{aligned}$$

3. Solve the transmission problem

$$\begin{aligned}
u_{xx} &= u_{tt} \quad x<0\ ,t\geq 0 \\
u_{xx} &= 4u_{tt} \quad x>0\ ,t\geq 0 \\
u(0-,t) &= u(0+,t) \\
u_x(0-,t) &= u_x(0+,t)
\end{aligned}$$

with the initial conditions

$$\begin{aligned}
&u(x,0)=\cos x\ ;\ u_t(x,0)=\sin x\ ;x<0 \\
&u(x,0)=0\ ;\ u_t(x,0)=0\ ;x>0
\end{aligned}$$

Ans.
For $x<-t$, we have D'Alembert's solution

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(\cos(x+t)+\cos(x-t))+\frac{1}{2}\int_{x-t}^{x+t}\sin s\,ds \\
&= \frac{1}{2}(\cos(x+t)+\cos(x-t))-\frac{1}{2}(\cos(x+t)-\cos(x-t)) \\
&= \cos(x-t)
\end{aligned}$$

For $-t<x<0$,

$$\begin{aligned}
u(x,t) &= \cos(x-t)+\phi(x+t) \\
u(0-,t) &= \cos t+\phi(t) \\
u_x(x,t) &= -\sin(x-t)+\phi'(x+t) \\
u_x(0-,t) &= \sin t+\phi'(t)
\end{aligned}$$

For $x>\frac{1}{2}t$ $u(x,t)=0$.

For $0 < x < \frac{1}{2}t$,

$$\begin{aligned} u(x,t) &= \psi(t-2x) \\ u(0+,t) &= \psi(t) \\ u_x(x,t) &= -2\psi'(t-2x) \\ u_x(0+,t) &= -2\psi\text{"}(t) \end{aligned}$$

Across the interface,

$$\begin{aligned} \sin t + \phi'(t) &= -2\psi'(t) \\ -\cos t + \phi(t) &= -2\psi(t) \\ \cos t + \phi(t) &= \psi(t) \\ \cos t + 3\phi(t) &= 0 \\ \phi(t) &= -\frac{1}{3}\cos t \\ 2\cos t &= 3\psi(t) \\ \psi(t) &= \frac{2}{3}\cos t \end{aligned}$$

Therefore, for $-t < x < 0$,

$$u(x,t) = \cos(x-t) - \frac{1}{3}\cos(x+t)$$

and for $0 < x < \frac{1}{2}t$,

$$u(x,t) = \frac{2}{3}\cos(t-2x)$$