MATH 3403
WEEK 9

The Heat Equation.

In the classification of linear second-order partial differential equations with two independent variables we met the class of *parabolic* equations intermediate between hyperbolic equations, which have two independent real characteristic variables, and elliptic equations which have no real characteristics.

The standard equation of this type is the *heat equation*

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}\ .$$

This equation has a wide variety of applications, some of which are
1. Conduction of heat in bars;
2. Diffusion of neutrons in atomic piles;
3. Evolution of probability distributions in random processes.

The equation has one family of characteristics, namely the lines t = constant. These lines can be considered as the limiting case of the characteristics $t \pm x/c =$ constant of the wave equation as the velocity c tends to infinity. In particular, the domain of dependence of the solution at the point (x_0, t_0) is the region $0 \le t < t_0$, and the range of influence is the half space $t > t_0$.

Heat conduction in a bar.

We assume that we are dealing with a uniform 'one-dimensional' bar, so that the governing equation is the heat equation above.

If the bar is of infinite length, then we only need to supply the initial temperature distribution $u(x, 0)$ in order to determine the temperature in the bar at future times.

If the bar is semi-infinite or finite we need to supply boundary conditions at the end(s) also. These are usually one of the three standard forms:
1. u given (assigned temperature) – the Dirichlet condition;
2. $\partial u/\partial x$ given (assigned heat flux, zero for insulated ends) – the Neumann condition;
3. $\partial u/\partial x + hu$ given (combined heat flux and radiation) – the Robin condition.

The Maximum Principle.

Common experience tells us that, in the absence of heat sources, the temperature in a bar or other conducting medium tends to even out over time. This is embodied in the following theorem.

`If` $u(x,t)$ `satisfies`

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}\ , \quad K > 0$$

`for` $0 \le x \le l$ `and` $0 \le t \le T$`, then the maximum value of` u `on` $[0, l] \times [0, T]$ `occurs either for` $t = 0$`, for` $x = 0$ `or for` $x = l$`.`

This property is referred to as the maximum principle for the heat equation.

Proof. Consider firstly a smooth function $v(x, t)$ which satifies the differential inequality

$$\frac{\partial v}{\partial t} < K\frac{\partial^2 v}{\partial x^2}$$

for $0 \le t \le T$ and $0 \le x \le l$.

(*We adopt this round-about approach since we require a strict inequality for the next step.*)

Suppose the v has a maximum at (x_0, t_0), where $0 < x_0 < l$, and $0 < t_0 \leq T$. Since we have a maximum,

$$v_{xx}(x_0, t_0) \leq 0 \ ,$$

and hence

$$v_t(x_0, t_0) < K v_{xx}(x_0, t_0) \leq 0$$

where the first inequality is strict. Since v is smooth, v_t is continuous, so that there is an interval $0 \leq t_0 - \delta \leq t \leq t_0$ throughout which v_t is negative. Then

$$v(x_0, t_0 - \delta) = v(x_0, t_0) - \int_0^{\delta} v_t(x_0, t_0 - s)\, ds > v(x_0, t_0) \ .$$

This contradicts our hypothesis that $v(x,t)$ has a maximum at (x_0, t_0). Hence v satisfies the maximum principle.

Now consider a solution $u(x,t)$ of the heat equation with $u \leq M$ for $t = 0$ or $x = 0$ or $x = l$. For any positive ϵ construct $v = u + \epsilon x^2$. Then

$$v_t - K v_{xx} = u_t - K u_{xx} - 2\epsilon K < 0$$

on $[0, l] \times [0, T]$. Therefore v takes its maximimum on $t = 0$ or $x = 0$ or $x = l$. i.e. $v \leq M + \epsilon l^2$. Since $u \leq v$ on $[0, l] \times [0, T]$, we have $u \leq M + \epsilon l^2$ in this region also, and since $\epsilon > 0$ is arbitrary,

$$u \leq M$$

throughout the region. QED

Consequences.

As in the case of Laplace's equation, the maximum principle implies uniqueness and well-posedness.

Corollary 1. `A solution` $u(x,t)$ `of the equation`

$$u_t = K u_{xx} + f(x,t)$$

`in the strip` $0 < x < l$, $t > 0$, `subject to the initial condition` $u(x, 0) = \phi(x)$ `and the boundary conditions` $u(0,t) = \mu_1(t)$, $u(l,t) = \mu_2(t)$ `is uniquely determined.`

Proof.

Suppose that $u_1(x,t)$ and $u_2(x,t)$ are two such solutions. Then $u(x,t) = u_1(x,t) - u_2(x,t)$ satisfies

$$u_t = K u_{xx} \ , \ u(x, 0) = 0 \ , \ u(0,t) = u(l,t) = 0$$

on $[0, l] \times [0, T]$ for every $T > 0$. Hence $u \leq 0$ in this region. Similarly $-u \leq 0$ on $[0, l] \times [0, T]$, so that $u = 0$ for $0 \leq x \leq l$, $t \geq 0$, and the two solutions are identical.

Note that this result does not guarantee the existence of a solution, merely its uniquness should it exist.

Corollary 2. The solution $u(x,t)$ of the initial and boundary problem depends continuously on the initial/boundary data.

Proof.

Suppose that u satisfies the equation above with the given initial/boundary data, and that v satisfies the same equation with the initial/boundary data

$$v(x,0) = \psi(x) \ , \ v(0,t) = \nu_1(t) \ , \ v(l,t) = \nu_2(t)$$

where

$$|\phi - \psi| < \epsilon \ , \ |\mu_1 - \nu_1| < \epsilon \ , \ |\mu_2 - \nu_2| < \epsilon \ .$$

Then $u - v$ satisfies the homogeneous heat equation,

$$(u-v)_t = K(u-v)_{xx}$$

with the initial data $(u-v)(x,0) = \phi(x) - \psi(x)$ and the boundary data $(u-v)(0,t) = \mu_1(t) - \nu_1(t)$ and $(u-v)(l,t) = \mu_2(t) - \nu_2(t)$, so that by the maximum principle

$$-\epsilon < u - v < \epsilon \ , \ 0 \le x \le l \ , \ t > 0 \ .$$

This shows that small variations in the initial/boundary data are not magnified in the solution, and that the finite problem with Dirichlet boundary data is well-posed.

Fourier Integral Transforms

Heat flow problems on a finite interval $[0,l]$ can be solved by means of the method of separation of variables. To find solutions on semi-infinite and infinite intervals we need to introduce the concept of the Fourier integral transforms. These transforms are generalizations of the full-range and half-range Fourier series.

For a function $f(x)$ defined on an interval $[-L,L]$, we have the Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(y) \cos\left(\frac{n\pi y}{L}\right) dy \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(y) \sin\left(\frac{n\pi y}{L}\right) dy \ .$$

Combining these we obtain

$$\begin{aligned} f(x) &= \frac{1}{2L}\int_{-L}^{L} f(y)\, dy \\ &\quad + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(y) \left(\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi y}{L}\right) + \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)\right) dy \\ &= \frac{1}{2L}\int_{-L}^{L} f(y)\, dy + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(y)\cos\left(\frac{n\pi(x-y)}{L}\right) dy \\ &= \frac{1}{2L}\sum_{-\infty}^{\infty}\int_{-L}^{L} f(y)\exp\left(i\frac{n\pi(x-y)}{L}\right) dy \end{aligned}$$

Setting $h = \pi/L$, we obtain

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\left\{h\sum_{-\infty}^{\infty}\int_{-L}^{L} f(y)\exp(i(x-y)(nh))\,dy\right\} \\ &= \frac{1}{2\pi}\sum_{-\infty}^{\infty} hF(nh) \end{aligned}$$

where

$$F(\omega) = \int_{-L}^{L} f(y)\exp(i(x-y)\omega)\,dy\ .$$

Now,

$$\sum_{-N}^{N} hF(nh) \sim \int_{-Nh}^{Nh} F(\omega)\,d\omega\ ,$$

so that

$$\sum_{-\infty}^{\infty} hF(nh) \sim \int_{-\infty}^{\infty} F(\omega)\,d\omega\ ,$$

an approximation which improves as $h \to 0$.

Letting $L \to \infty$ (and therefore $h \to 0$), we obtain

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\left(\int_{-\infty}^{\infty} f(y)\exp i(x-y)\omega\,dy\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega x}\left(\int_{-\infty}^{\infty} dy\, e^{-i\omega y} f(y)\right) \end{aligned}$$

The pair of equations

$$\begin{aligned} \mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(y)e^{-i\omega y}\,dy &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \\ f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(\omega)e^{i\omega x}\,d\omega & \end{aligned}$$

are known as the Fourier Transform pair.

The first equation defines the Fourier Transform of the function f, while the second defines the inverse transform. This is the form in which the transform usually appears, and which will be used in this course. However, you should be aware that some authors prefer the symmetric definition

$$\begin{aligned} \hat{\mathcal{F}}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \\ f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\mathcal{F}}(\omega)e^{i\omega x}\,d\omega \end{aligned}$$

(This form can be recognised in practice by the extraneous factors of $\sqrt{2\pi}$.)

Properties of the Fourier Transform.

The Fourier Transform is used in much the same way as the Laplace Transform, to which it is related.

While the Laplace Transform is used to solve initial value problems on a semi-infinite (time) domain, the Fourier Transform is used to simplify the operator $\partial^2/\partial x^2$ on infinite domains.

Note that if the Fourier transform of f is defined for $\omega \in \mathbb{R}$, then in particular

$$\int_{-\infty}^{\infty} f(x)\,dx \quad \text{is finite,}$$

so that we have

$$\lim_{x\to\infty} f(x) = 0 \quad \text{and} \quad \lim_{x\to-\infty} f(x) = 0\ .$$

As with the Laplace transform, the principal applications of the Fourier transform depend on the way it affects derivatives.

If

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

then

$$\begin{aligned}
\int_{-\infty}^{\infty} f'(x)e^{-i\omega x}\,dx &= \ f(x)e^{-i\omega x}\Big|_{-\infty}^{\infty} + i\omega\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \\
&= i\omega\mathcal{F}(\omega) \\
\int_{-\infty}^{\infty} f''(x)e^{-i\omega x}\,dx &= \ -\omega^2\mathcal{F}(\omega)
\end{aligned}$$

Note the similarity to the Laplace transform in that the transform of the derivative is a multiple of the transform of the function, so that the transform replaces the appropriate differential operators by algebraic expressions. Unlike the Laplace transform there are no additional constant terms in the transforms of the derivatives. This is because the range of integration is $(-\infty, \infty)$.

Consider for example the heat equation

$$\begin{aligned}
u_t = Ku_{xx}\ ,\ -\infty < x < \infty\ , t \geq 0 \\
u(x,0) = f(x)\ ,\ -\infty < x < \infty
\end{aligned}$$

If we define

$$\mathcal{U}(\omega,t) = \int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx$$

then

$$\begin{aligned}
\int_{-\infty}^{\infty} u_t(x,t)e^{-i\omega x}\,dx &= K\int_{-\infty}^{\infty} u_{xx}(x,t)e^{-i\omega x}\,dx \\
\frac{\partial}{\partial t}\mathcal{U}(\omega,t) &= \ -K\omega^2\mathcal{U}(\omega,t) \\
\mathcal{U}(\omega,t) &= \mathcal{U}(\omega,0)e^{-K\omega^2 t}
\end{aligned}$$

From the initial data we have

$$\mathcal{U}(\omega,0)=\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx=\mathcal{F}(\omega)\quad\text{say },$$

so that

$$\mathcal{U}(\omega,t)=\mathcal{F}(\omega)e^{-K\omega^2 t}$$

and the required solution is obtained by finding the inverse Fourier transform of the function on the right hand side.

The Convolution Theorem.

The form we have found above is the product of two functions of ω, one at least of which is a Fourier transform. When we have the product of two Fourier transforms, we can express this as the Fourier transform of an integral.

Suppose

$$\begin{aligned}
\mathcal{F}(\omega) &= \int_{-\infty}^{\infty} f(\xi)e^{-i\omega\xi}\,d\xi\\
\text{and}\qquad \mathcal{G}(\omega) &= \int_{-\infty}^{\infty} g(y)e^{-i\omega y}\,dy\\
\text{then}\qquad \mathcal{F}(\omega)\mathcal{G}(\omega) &= \int_{-\infty}^{\infty} d\xi\,\left(\int_{-\infty}^{\infty} e^{-i\omega(\xi+y)}f(\xi)g(y)\,dy\right)\\
&= \int_{-\infty}^{\infty} d\xi\int_{-\infty}^{\infty} e^{-i\omega x}f(\xi)g(x-\xi)\,dx\\
&= \int_{-\infty}^{\infty} dx\,e^{-i\omega x}\left(\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi\right)
\end{aligned}$$

The integral

$$\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi$$

is called the convolution of the functions f and g.

The Fundamental Solution of the heat equation.

The function whose Fourier transform is $\exp(-K\omega^2 t)$ is

$$\begin{aligned}
g(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-K\omega^2 t+i\omega x}\,d\omega\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\bigl(-Kt(\omega^2-(ix/Kt)\omega)\bigr)\,d\omega\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\bigl(-Kt(\omega-(ix/2Kt))^2-x^2/4Kt\bigr)\,d\omega\\
&= \frac{1}{2\pi}e^{-x^2/4Kt}\int_{-\infty}^{\infty} e^{-Kt\xi^2}\,d\xi\\
&= \frac{1}{2\pi}\frac{1}{\sqrt{Kt}}e^{-x^2/4Kt}\int_{-\infty}^{\infty} e^{-y^2}\,dy\\
&= \frac{1}{2\sqrt{\pi Kt}}e^{-x^2/4Kt}
\end{aligned}$$

This solution is known as the *Fundamental Solution* of the heat equation. Note that it is not defined for $t < 0$ and that its limit as $t \to 0+$ is 0 except for $x = 0$, when the limit is ∞. This function represents the solution of the heat equation when the initial heat distribution is $\delta(x)$.

Using this solution and the convolution theorem we obtain the general solution of the heat equation in an infinite bar as

$$u(x,t) = \frac{1}{2\sqrt{\pi K t}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4Kt} f(\xi)\, d\xi \ .$$

Note that the solution for (x,t), $t > 0$, depends on $f(\xi)$ for $-\infty < \xi < \infty$, confirming that the domain of dependence is infinite in extent.