MATH 3403

WEEK 8

Dirichlet's Principle.

In this section we will consider the more general second order elliptic equation

$$\nabla^2 u = P(\boldsymbol{x})u$$

in some finite region $\mathcal{R}$, where P is continuous on $\mathcal{R}$.

Starting with the Green's Identity

$$\iiint_{\mathcal{R}} \left((\nabla u)^2 + u\nabla^2 u\right)\, dV = \iint_{\partial\mathcal{R}} u\frac{\partial u}{\partial n}\, dS$$

we see that if u is a solution of the equation on $\mathcal{R}$, then

$$\iiint_{\mathcal{R}} \left((\nabla u)^2 + Pu^2\right)\, dV = \iint_{\partial\mathcal{R}} u\frac{\partial u}{\partial n}\, dS$$

The expression on the left of this equation is referred to as the **Dirichlet Integral** associated with this equation.

If $P \geq 0$ on $\mathcal{R}$, and P is not identically zero, then this integral is positive definite; that is

$$||u||^2 = \iiint_{\mathcal{R}} \left((\nabla u)^2 + Pu^2\right)\, dV \;\; > 0$$

if $u \neq 0$.

In this case the solutions of the various boundary value problems:

The Dirichlet Problem

$$\nabla^2 u = P(\boldsymbol{x})u \; ; \; u = f \text{ on } \partial\mathcal{R}$$

The Neumann Problem

$$\nabla^2 u = P(\boldsymbol{x})u \; ; \; \frac{\partial u}{\partial n} = f \text{ on } \partial\mathcal{R}$$

and the Robin Problem

$$\nabla^2 u = P(\boldsymbol{x})u \; ; \; u + \alpha\frac{\partial u}{\partial n} = f \text{ on } \partial\mathcal{R} \; ; \alpha > 0$$

are all unique if they exist. For if u_1 and u_2 are solutions, then their difference $u = u_1 - u_2$ is a solution of of the equation satisfying homogeneous boundary data, so that for the Dirichlet and Neumann problems we have

$$||u||^2 = \iint_{\partial\mathcal{R}} u\frac{\partial u}{\partial n}\, dS = 0$$

so that $u = 0$ in $\mathcal{R}$, while for the Robin problem we have

$$||u||^2 = -\alpha^{-1} \iint_{\partial\mathcal{R}} u^2\, dS \leq 0$$

so that $u = 0$ in $\mathcal{R}$ and on $\partial\mathcal{R}$.

When $P = 0$ on $\mathcal{R}$, we have Laplace's equation, for which the solutions of the Dirichlet and Robin problems are unique, but, as we have seen, the solution of the Neumann problem involves an arbitrary constant and there is a consistency condition on the boundary data in this case.

If P takes negative values on $\mathcal{R}$ there may be multiple solutions for these boundary value problems. However, we will only consider some specialised forms of this case.

For the time being we will assume that $P \geq 0$, and that we wish to solve a Dirichlet problem on $\mathcal{R}$.

Associated with the Dirichlet integral there is an inner product

$$(u, v) = \iiint_{\mathcal{R}} (\nabla u \nabla v + Puv)\, dV \ .$$

Note that $(u, u) = ||u||^2$, and that $(u, v) = (v, u)$.

Using the Green's Identity

$$\iiint_{\mathcal{R}} \nabla u \nabla v \, dV = \iint_{\partial\mathcal{R}} v \frac{\partial u}{\partial n} \, dS - \iiint_{\mathcal{R}} v \nabla^2 u \, dV$$

we get

$$(u, v) = \iiint_{\mathcal{R}} (-\nabla^2 u + Pu) v \, dV + \iint_{\partial\mathcal{R}} v \frac{\partial u}{\partial n} \, dS$$

Therefore, if u is any solution of $\nabla^2 u = Pu$, and $v = 0$ on $\partial\mathcal{R}$, then $(u, v) = 0$.

Suppose that we are trying to solve the Dirichlet problem

$$\nabla^2 u = Pu \text{ on } \mathcal{R} \ ; \ u = f \text{ on } \partial\mathcal{R}$$

If w is any sufficiently well-behaved function satisfying $w = f$ on $\partial\mathcal{R}$, and u is the solution (which we assume exists), then $v = w - u$ vanishes on $\partial\mathcal{R}$. Therefore

$$\begin{aligned} ||w||^2 &= (w, w) \\ &= (u + v, u + v) \\ &= (u, u) + 2(u, v) + (v, v) \\ &= ||u||^2 + ||v||^2 \\ &\geq ||u||^2 \end{aligned}$$

This means that, of all the suitable functions satisfying the boundary conditions, the solution u is the one which minimizes the value of the Dirichlet integral. This result is known as Dirichlet's principle.

(It is by no means obvious that such a minimizing function exists! Dirichlet assumed that it was so, but the existence was only proved many years later by Hilbert.)

Applications - the Galerkin method.

This result is the basis for many approximate solution methods for elliptic partial differential equations.

For example, we can consider some (finite) family of functions $\{v_n\}$ which vanish on $\mathcal{R}$, and consider the class of functions

$$w = w_0 + \sum_{i=1}^{n} a_i v_i$$

where w_0 satisfies the boundary conditions.

Substituting into the Dirichlet integral gives

$$\begin{aligned} ||w||^2 = & \left(w_0 + \sum_{i=1}^{n} a_i v_i, w_0 + \sum_{j=1}^{n} a_j v_j \right) \\ & = (w_0, w_0) + \sum_{i=1}^{n} a_i (v_i, w_0) + \sum_{j=1}^{n} a_j (w_0, v_j) + \sum_{i=1}^{n} \sum_{j=1}^{n} a_i (v_i, v_j) a_j \\ & = ||w_0||^2 + 2 \sum_{i=1}^{n} b_i a_i + \sum_{i=1}^{n} \sum_{j=1}^{n} a_i c_{ij} a_j \end{aligned}$$

where $b_i = (w_0, v_i)$ and $c_{ij} = (v_i, v_j) = c_{ji}$.

Taking the partial derivatives with respect to the a_i and setting them to 0, we obtain the system of equations

$$\sum_{j=1}^{n} c_{ij} a_j + b_i = 0$$

$$\left(v_i, \sum_{j=1}^{n} a_j v_j \right) + (v_i, w_0) = 0$$

$$(v_i, w) = 0 \ ; \ i = 1, \dots n$$

which we solve to find the co-ordinates of the stationary value.

Since the Dirichlet integral is positive definite, this stationary value yields a minimium, and the corresponding sum is an approximate solution for the Dirichlet problem.

For example, consider the Dirichlet problem

$$u'' = u \ ; \ u(-1) = u(1) = 1 \ .$$

We take $w_0 = 1$, and consider $v_1 = (1 - x^2)$ and $v_2 = (1 - x^4)$.

The inner product is

$$(u, v) = \int_{-1}^{1} (u'v' + uv) \, dx \ .$$

$$\begin{aligned} ||v_1||^2 = & \int_{-1}^{1} ((-2x)(-2x) + (1 - x^2)^2) \, dx \\ = & \int_{-1}^{1} (4x^2 + 1 - 2x^2 + x^4) \, dx \\ = & \ x + \frac{2}{3} x^3 + \frac{1}{5} x^5 \Big|_{-1}^{1} \\ = & \ 2 + \frac{4}{3} + \frac{2}{5} = \frac{56}{15} \end{aligned}$$

$$\begin{aligned}
||v_2||^2 &= \int_{-1}^{1}((-4x^3)(-4x^3)+(1-x^4)^2)\,dx \\
&= \int_{-1}^{1}(16x^6+1-2x^4+x^8)\,dx \\
&= x-\frac{2}{5}x^5+\frac{16}{7}x^7+\frac{1}{9}x^9\Big|_{-1}^{1} \\
&= 2-\frac{4}{5}+\frac{32}{7}+\frac{2}{9}=\frac{1888}{315}
\end{aligned}$$

$$\begin{aligned}
(v_1,v_2) &= \int_{-1}^{1}((-2x)(-4x^3)+(1-x^2)(1-x^4))\,dx \\
&= \int_{-1}^{1}(8x^4+1-x^2-x_4+x^6)\,dx \\
&= x-\frac{1}{3}x^3+\frac{7}{5}x^5+\frac{1}{7}x^7\Big|_{-1}^{1} \\
&= 2-\frac{2}{3}+\frac{14}{5}+\frac{2}{7}=\frac{464}{105}
\end{aligned}$$

$$\begin{aligned}
(w_0,v_1) &= \int_{-1}^{1}((0)(-2x)+1(1-x^2))\,dx \\
&= x-\frac{1}{3}x^3\Big|_{-1}^{1} \\
&= 2-\frac{2}{3}=\frac{4}{3}
\end{aligned}$$

$$\begin{aligned}
(w_0,v_2) &= \int_{-1}^{1}((0)(-4x^3)+1(1-x^4))\,dx \\
&= x-\frac{1}{5}x^5\Big|_{-1}^{1} \\
&= 2-\frac{2}{5}=\frac{8}{5}
\end{aligned}$$

As a first approximation, we consider $u=w_0+a_1v_1$.
a_1 is the solution of

$$\begin{gathered}
||v_1||^2a_1+(w_0,v_1)=0 \\
\frac{56}{15}a_1+\frac{4}{3}=0 \\
a_1=-\frac{5}{14} \\
u\sim 1-\frac{5}{14}(1-x^2)=.643+.357x^2
\end{gathered}$$

If we can compare this result with the exact solution

$$u = \frac{\cosh x}{\cosh 1} = .648 + .324x^2 + \dots$$

we see that this approximate solution give 2dp accuracy over $[-1.1]$.

A better approximation considers $u = w_0 + a_1 v_1 + a_2 v_2$.

The coefficients a_1 and a_2 are found by solving simultaneously the equations

$$\begin{aligned}
||v_1||^2 a_1 + (v_1, v_2) a_2 &= -(w_0, v_1) \\
(v_1, v_2) a_1 + ||v_2||^2 &= -(w_0, v_2) \\
\frac{56}{15} a_1 + \frac{464}{105} a_2 &= -\frac{4}{3} \\
\frac{464}{105} a_1 + \frac{1888}{315} a_2 &= -\frac{8}{5} \\
a_1 = -\frac{119}{368} &= -0.32337 \\
a_2 = -\frac{21}{736} &= -0.02853
\end{aligned}$$

giving

$$u \sim 1 - \frac{119}{368}(1 - x^2) - \frac{21}{736}(1 - x^4) = 0.64810 + 0.32337x^2 + 0.02853x^4$$

compared to the exact solution

$$u = 0.64805 + 0.32402x^2 + 0.02700x^4 + \dots$$

These calculations show that in general we need to recalculate all the coefficients whenever we increase the number of terms in our expansion.

However, this difficulty can be avoided by choosing an orthogonal set of functions; that is, functions for which $(v_i, v_j) = 0$ when $i \neq j$, since in this case the matrix (c_{ij}) is diagonal, and the equations have the solutions $a_i = -(w_0, v_i)/||v_i||^2$ for each i.

For example, for this problem we could choose

$$v_i = \cos\left(\left(i + \frac{1}{2}\right)\pi x\right) .$$

$$\begin{aligned}
(v_i, v_j) = & \int_{-1}^{1} [(i + 0.5)\pi \sin((i + 0.5)\pi x)(j + 0.5)\pi \sin((j + 0.5)\pi x) \\
& + \cos((i + 0.5)\pi x) \cos((j + 0.5)\pi x)]\, dx \\
& = \left(i + \frac{1}{2}\right)\left(j + \frac{1}{2}\right)\pi^2 \frac{1}{2} \int_{-1}^{1} [\cos((i - j)\pi x) - \cos((i + j + 1)\pi x)]\, dx \\
& \quad + \frac{1}{2} \int_{-1}^{1} [\cos(i - j)\pi x] + \cos((i + j + 1)\pi x)]\, dx \\
& = 0 \quad \text{if } i \neq j
\end{aligned}$$

When $i = j$,

$$||v_i||^2 = \left(i + \frac{1}{2}\right)^2 \pi^2 + 1 \ .$$

$$\begin{aligned}(w_0, v_i) &= \int_{-1}^{1} \cos\left(\left(i + \frac{1}{2}\right)\pi x\right)\, dx \\ &= \frac{2}{(2i+1)\pi} \sin\left(\left(i + \frac{1}{2}\right)\pi x\right)\Big|_{-1}^{1} \\ &= (-1)^i \frac{4}{(2i+1)\pi}\end{aligned}$$

The simultaneous equations reduce to

$$||v_i||^2 a_i + (w_0, v_i) = 0$$

so that we can calculate successive approximations without having to recalculate the preceding coefficients.

Indeed, since we can in this case determine

$$a_i = (-1)^{i+1} \frac{16}{\pi((2i+1)^3\pi^2 + 8i + 4)}$$

we can express the solution as

$$u = 1 + \sum_{i=0}^{\infty} (-1)^{i+1} \frac{16}{\pi((2i+1)^3\pi^2 + 8i + 4)} \cos\left(\left(i + \frac{1}{2}\right)\pi x\right)$$

This series certainly converges uniformly on $[-1, 1]$ since it is dominated by

$$\frac{16}{\pi^3} \sum_{i=0}^{\infty} \frac{1}{(2i+1)^3} \ .$$

However this does not guarantee that it converges to the solution.

In order to ensure convergence to the solution, we need to know that the set of functions $\{v_i\}$ is **complete**; that is, that any solution can be expressed solely in terms of these functions.

In fact, in this case the complete set of such functions also includes the functions $\sin(i\pi x)$ for $i = 1$ to ∞. These functions are however odd on the interval $[-1, 1]$ and hence do not appear in the expansion of the solution which is even. If the boundary data were not even, it would be necessary to include them in the calculations.

Eigenfunctions.

We are looking for non-trivial functions v_i such that $(v_i, v_j) = 0$ when $i \neq j$, and $v_i = 0$ on $\partial\mathcal{R}$.

$$\begin{aligned}(v_i, v_j) &= \iiint_{\mathcal{R}} (-\nabla^2 v_i + P v_i) v_j \, dV + \iint_{\partial\mathcal{R}} v_j \frac{\partial v_i}{\partial n}\, dS \\ &= \iiint_{\mathcal{R}} (-\nabla^2 v_i + P v_i) v_j \, dV\end{aligned}$$

Suppose that

$$-\nabla^2 v_i + Pv_i = \lambda_i w(\boldsymbol{x}) v_i$$
$$-\nabla^2 v_j + Pv_j = \lambda_j w(\boldsymbol{x}) v_j \ \lambda_i \neq \lambda_j$$

where w is a fixed positive function. In particular, we could take $w = P$. Then

$$(v_i, v_j) = \lambda_i \iiint_{\mathcal{R}} w(\boldsymbol{x}) v_i v_j \, dV$$
$$(v_j, v_i) = \lambda_j \iiint_{\mathcal{R}} w(\boldsymbol{x}) v_j v_i \, dV$$
$$(v_i, v_j) = 0 \ ; \ i \neq j$$

(If $\lambda_i = \lambda_j$ but $v_i \neq cv_j$, we can choose $v_j^* = v_j - cv_i$ so that $(v_i, v_j^*) = 0$.)

Therefore a suitable set of functions are the eigenfunctions; non-trivial solutions of

$$\nabla^2 v = (P - \lambda w(\boldsymbol{x}))v \text{ in } \mathcal{R}$$
$$v = 0 \text{ on } \partial\mathcal{R}$$

We have already seen that this equation has only the zero solution if $P - \lambda w \geq 0$ on $\mathcal{R}$. Therefore any eigenvalues will be positive. We will return to this topic later.

Consider the Dirichlet problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \begin{cases} 0 < x < 1 \\ 0 < y < 1 \end{cases}$$
$$u(0, y) = 0 \ ; \ u(1, y) = y(1 - y)$$
$$u(x, 0) = 0 \ ; \ u(x, 1) = 0$$

We have already seen how to obtain a solution for this problem

$$\left(u = \frac{8}{\pi^3} \sum_{k=0}^{\infty} \frac{\sinh(2k+1)\pi x \, \sin(2k+1)\pi y}{(2k+1)^3 \sinh(2k+1)\pi} \right)$$

by the method of separation of variables.

As an alternative, we note that;

(a) $w_0 = xy(1 - y)$ satisfies the boundary conditions;

(b) the functions $v_{ij} = \sin i\pi x \, \sin j\pi y$ satisfy

$$\nabla^2 v_{ij} = -(i^2 + j^2)\pi^2 v_{ij} \ ,$$

and vanish on the boundary.

Therefore, if we set

$$u = w_0 + \sum_{i,j} a_{ij} v_{ij} \ ,$$

Galerkin's method gives

$$(w_0, v_{ij}) + a_{ij} ||v_{ij}||^2 = 0$$

where

$$(u,v) = \int_0^1 \int_0^1 \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) dx\,dy\ .$$

$$\begin{aligned}
||v_{ij}||^2 = & \int_0^1 \int_0^1 \left(\left(\frac{\partial v_{ij}}{\partial x}\right)^2 + \left(\frac{\partial v_{ij}}{\partial y}\right)^2 \right) dxdy \\
= & \int_0^1 \int_0^1 i^2\pi^2 \cos^2(i\pi x)\sin^2(i\pi y)\,dxdy \\
& + \int_0^1 \int_0^1 j^2\pi^2 \sin^2(i\pi x)\cos^2(j\pi y)\,dxdy \\
= & \, i^2\pi^2 \int_0^1 \cos^2(i\pi x)\,dx \int_0^1 \sin^2(j\pi y)\,dy \\
& + j^2\pi^2 \int_0^1 \sin^2(i\pi x)\,dx \int_0^1 \cos^2(j\pi y)\,dy \\
= & \, \frac{\pi^2}{4}(i^2+j^2)
\end{aligned}$$

$$\begin{aligned}
(w_0, v_{ij}) = & \int_0^1 i\pi\cos(i\pi x)\,dx \int_0^1 y(1-y)\sin(j\pi x)\,dy \\
& + \int_0^1 x\sin(i\pi x)\,dx \int_0^1 j\pi(1-2y)\cos(j\pi x)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \cos(i\pi x)\,dx &= 0 \\
\int_0^1 x\sin(i\pi x)\,dx = & \; -\frac{1}{i\pi}x\cos(i\pi x)\Big|_0^1 + \frac{1}{i\pi}\int_0^1 \cos(i\pi x)\,dx \\
= & \; \frac{(-1)^{i+1}}{i\pi} \\
\int_0^1 (1-2y)\cos(j\pi y)\,dy = & \; \frac{1}{j\pi}(1-2y)\sin(j\pi y)\Big|_0^1 + \frac{2}{j\pi}\int_0^1 \sin(j\pi y)\,dy \\
= & \; -\frac{2}{j^2\pi^2}\cos(j\pi y)\Big|_0^1 = \frac{2}{j^2\pi^2}(1-(-1)^j) \\
(w_0, v_{ij}) = & \; (-1)^{i+1}\frac{2}{ij\pi^2}(1-(-1)^j)
\end{aligned}$$

Combining these results we obtain the expansion

$$u = xy(1-y) + \frac{16}{\pi^4}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^i \sin(i\pi x)\sin((2k+1)\pi y)}{i(2k+1)(i^2+(2k+1)^2)}$$

for the solution.

This double sum converges to the same solution as the expansion given by the separation of variables method.

However, if we truncate the series (as is usual) the separation of variables approach gives a function which satisfies the differential equation exactly but approximates the boundary values, while the Galerkin method gives a function which satisfies the boundary data exactly but is only approximately a solution of the differential equation.

Functions with local support.

In general it is difficult (impossible) to determine a family of eigenfunctions for an arbitrary domain and equation. Instead, it is usual to choose the functions v_i so that they are zero everywhere outside some small region. This means that the derivatives are also zero outside this region, and (v_i, v_j) reduces to an integral over the region in which the functions v_i and v_j are simultaneously non-zero. A judicious choice of functions means that most of the coefficients c_{ij} in the system of equations which determine the coefficients a_j will be zero, which simplifies the computational procedures.

For example, in one dimension we can split the interval $[a, b]$ into n equal intervals of width $h = (b - a)/n$, and define a set $\{v_i\}$ of $n - 1$ functions such that

$$v_i = \begin{cases} 0 & x < a + (i-1)h \\ \frac{1}{h}(x - (a + (i-1)h) & a + (i-1)h \le x \le a + ih \\ \frac{1}{h}(a + (i+1)h - x) & a + ih < x \le a + (i+1)h \\ 0 & x > a + (i+1)h \end{cases}$$

The sum $\sum a_i v_i$ represents a function which vanishes at a and b, takes the value a_i at $a + ih$ and is straight between these points.

For these functions, $(v_i, v_j) = 0$ unless $j = i - 1, i$ or $i + 1$.

The Galerkin equations therefore reduce to a tridiagonal system

$$(v_i, v_{i-1})a_{i-1} + (v_i, v_i)a_i + (v_i, v_{i+1})a_{i+1} = -(w_0, v_i) \ .$$

For example, suppose we take our equation

$$u'' = u \ ; \ u(-1) = u(1) = 1$$

again, and split the interval $[-1, 1]$ into eight segments.

A typical function v has the form

$$v = \begin{cases} 4\left(x + \frac{1}{4}\right) & ; -\frac{1}{4} \le x \le 0 \\ 4\left(\frac{1}{4} - x\right) & ; 0 \le x \le 14 \end{cases}$$

$$\begin{aligned} ||v_i||^2 &= \int_{-\frac{1}{4}}^{\frac{1}{4}} 4^2 \, dx + \int_{-\frac{1}{4}}^{0} 16 \left(x + \frac{1}{4}\right)^2 dx + \int_0^{\frac{1}{4}} 16 \left(x - \frac{1}{4}\right)^2 dx \\ &= 8 + \frac{16}{3}\frac{1}{4^3} + \frac{16}{3}\frac{1}{4^3} \\ &= 8\frac{1}{6} \end{aligned}$$

$$\begin{aligned}(v_i, v_{i+1}) &= \int_0^{\frac{1}{4}} (-16)\, dx + \int_0^{\frac{1}{4}} 16x\left(\frac{1}{4} - x\right)\, dx \\ &= -4 + 2\frac{1}{4^2} - \frac{16}{3}\frac{1}{4^3} \\ &= -3\frac{23}{24}\end{aligned}$$

$$\begin{aligned}(w_0, v_i) &= \int_{-\frac{1}{4}}^{0} (4x+1)\, dx + \int_0^{\frac{1}{4}} (1-4x)\, dx \\ &= -2\frac{1}{4^2} + \frac{1}{4} + \frac{1}{4} - 2\frac{1}{4^2} \\ &= \frac{1}{4}\end{aligned}$$

which leads to the system of equations

$$\begin{pmatrix} \frac{49}{6} & -\frac{95}{24} & 0 & 0 & 0 & 0 & 0 \\ -\frac{95}{24} & \frac{49}{6} & -\frac{95}{24} & 0 & 0 & 0 & 0 \\ 0 & -\frac{95}{24} & \frac{49}{6} & -\frac{95}{24} & 0 & 0 & 0 \\ 0 & 0 & -\frac{95}{24} & \frac{49}{6} & -\frac{95}{24} & 0 & 0 \\ 0 & 0 & 0 & -\frac{95}{24} & \frac{49}{6} & -\frac{95}{24} & 0 \\ 0 & 0 & 0 & 0 & -\frac{95}{24} & \frac{49}{6} & -\frac{95}{24} \\ 0 & 0 & 0 & 0 & 0 & -\frac{95}{24} & \frac{49}{6} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = - \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$$

whose solution is

$$-(0.1616,\, 0.2703,\, 0.3328,\, 0.3532,\, 0.3328,\, 0.2703,\, 0.1616)'$$

The reduced wave equation.

Consider the multidimensional wave equation

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

for some bounded region $\mathcal{R} \in \mathbb{R}^n$ and for $t \geq 0$, where for simplicity we will assume that $u(\boldsymbol{x}, t) = 0$ on $\partial\mathcal{R}$.

In two dimensions, for example, this represents a vibrating drum.

If we set $u = F(\boldsymbol{x})T(t)$, and separate the variables, we obtain the pair of equations

$$\begin{aligned}\nabla^2 F + \lambda F = 0 \\ T'' + \lambda T = 0\end{aligned}$$

where the eigenvalues λ are determined by the non-trivial solutions of

$$\nabla^2 F + \lambda F = 0\ ;\ F = 0 \text{ on } \mathcal{R}\ .$$

If we consider the Dirichlet integral,

$$\begin{aligned}||F||^2 &= \int_{\mathcal{R}} \left((\nabla F)^2 - \lambda F^2\right) dV \\ &= -\int_{\mathcal{R}} F\left(\nabla^2 F + \lambda F\right) dV + \int_{\partial\mathcal{R}} F\frac{\partial F}{\partial n}\, dS \\ &=0\end{aligned}$$

when the above equation is satisfied. Therefore, if λ is an eigenvalue and F is the corresponding eigenfunction

$$\begin{aligned}\int_{\mathcal{R}} (\nabla F)^2\, dV &= \lambda \int_{\mathcal{R}} F^2\, dV \\ \lambda &= \int (\nabla F)^2\, dV \Big/ \int F^2\, dV > 0\ .\end{aligned}$$

Since the eigenvalues are positive, the solutions of the second equation

$$T'' + \lambda T = 0$$

are of the form

$$T = a\cos(\sqrt{\lambda}t) + b\sin(\sqrt{\lambda}t)$$

and all the solutions oscillate.

Suppose that $G(\boldsymbol{x},\boldsymbol{\xi})$ is a Green's function for the region $\mathcal{R}$. The Poisson equation

$$\nabla^2 u = F \text{ in } \mathcal{R}\ ; u = 0 \text{ on } \mathcal{R}$$

has the solution

$$u(\boldsymbol{x}) = -\frac{1}{c_n}\int_{\mathcal{R}} G(\boldsymbol{x},\boldsymbol{\xi})F(\boldsymbol{\xi})\, d\boldsymbol{\xi}$$

where c_n is the appropriate dimensional scale factor.

Applying this result to the eigenfunction equation, we see that the eigenfunctions and eigenvalues are solutions of the integral equation

$$F(\boldsymbol{x}) = \lambda\frac{1}{c_n}\int_{\mathcal{R}} G(\boldsymbol{x},\boldsymbol{\xi})F(\boldsymbol{\xi})\, d\boldsymbol{\xi}$$

Suppose that λ_1 and λ_2 are distinct eigenvalues, with corresponding eigenfunctions ϕ_1 and ϕ_2. Then

$$\begin{aligned}\phi_1(\boldsymbol{x}) &= \frac{\lambda_1}{c_n}\int_{\mathcal{R}} G(\boldsymbol{x},\boldsymbol{\xi})\phi_1(\boldsymbol{\xi})\, d\boldsymbol{\xi} \\ \phi_2(\boldsymbol{x})\phi_1(\boldsymbol{x}) &= \frac{\lambda_1}{c_n}\int_{\mathcal{R}} \phi_2(\boldsymbol{x})G(\boldsymbol{x},\boldsymbol{\xi})\phi_1(\boldsymbol{\xi})\, d\boldsymbol{\xi} \\ \int_{\mathcal{R}} \phi_2(\boldsymbol{x})\phi_1(\boldsymbol{x})\, d\boldsymbol{x} &= \frac{\lambda_1}{c_n}\int_{\mathcal{R}}\int_{\mathcal{R}} \phi_2(\boldsymbol{x})G(\boldsymbol{x},\boldsymbol{\xi})\phi_1(\boldsymbol{\xi})\, d\boldsymbol{\xi}\, d\boldsymbol{x} \\ \int_{\mathcal{R}} \phi_1(\boldsymbol{x})\phi_2(\boldsymbol{x})\, d\boldsymbol{x} &= \frac{\lambda_2}{c_n}\int_{\mathcal{R}}\int_{\mathcal{R}} \phi_1(\boldsymbol{x})G(\boldsymbol{x},\boldsymbol{\xi})\phi_2(\boldsymbol{\xi})\, d\boldsymbol{\xi}\, d\boldsymbol{x} \\ &= \frac{\lambda_2}{c_n}\int_{\mathcal{R}}\int_{\mathcal{R}} \phi_2(\boldsymbol{x})G(\boldsymbol{x},\boldsymbol{\xi})\phi_1(\boldsymbol{\xi})\, d\boldsymbol{\xi}\, d\boldsymbol{x} \\ \int_{\mathcal{R}} \phi_1(\boldsymbol{x})\phi_2(\boldsymbol{x})\, d\boldsymbol{x} &= 0\end{aligned}$$

since $G(\boldsymbol{x},\boldsymbol{\xi}) = G(\boldsymbol{\xi},\boldsymbol{x})$ and $\lambda_1 \neq \lambda_2$.

Where we have repeated eigenvalues we can use a Gram-Schmidt process to obtain orthogonal eigenfunctions, which we can scale so that

$$\int_{\mathcal{D}} \phi^2(\boldsymbol{x})\, d\boldsymbol{x} = 1 \ .$$

If we assume that $G(\boldsymbol{x},\boldsymbol{\xi})$ can be expanded in the form $\sum a_i \phi(\boldsymbol{\xi})$, then the Fourier coefficients are given by

$$a_i = \int_{\mathcal{R}} G(\boldsymbol{x},\boldsymbol{\xi})\phi_i(\boldsymbol{\xi})\, d\boldsymbol{\xi} = \frac{c_n}{\lambda_i}$$

so that

$$G(\boldsymbol{x},\boldsymbol{\xi}) \sim c_n \sum \frac{\phi_i(\boldsymbol{x})\phi_i(\boldsymbol{\xi})}{\lambda_i}$$

More precisely, if we consider

$$\frac{1}{c_n} G(\boldsymbol{x},\boldsymbol{\xi}) - \sum_{i=1}^{m} \frac{\phi(\boldsymbol{x})\phi(\boldsymbol{\xi})}{\lambda_i}$$

for any finite set of eigenvalues and eigenfunctions, we have

$$\begin{aligned}
0 \ &\leq \int_{\mathcal{R}} \left(\frac{1}{c_n} G(\boldsymbol{x},\boldsymbol{\xi}) - \sum_{i=1}^{m} \frac{\phi_i(\boldsymbol{x})\phi_(\boldsymbol{\xi})}{\lambda_i} \right)^2 d\boldsymbol{\xi} \\
&= \frac{1}{c_n^2} \int_{\mathcal{R}} G^2\, d\boldsymbol{\xi} \\
&\quad - \frac{2}{c_n} \sum_{i=1}^{m} \frac{\phi_i(\boldsymbol{x})}{\lambda_i} \int_{\mathcal{R}} G(\boldsymbol{x},\boldsymbol{\xi})\phi_i(\boldsymbol{\xi})\, d\boldsymbol{\xi} \\
&\quad + \sum_{i=1}^{m} \frac{\phi_i(\boldsymbol{x})^2}{\lambda_i^2} \int_{\mathcal{D}} \phi(\boldsymbol{\xi})^2\, d\boldsymbol{\xi} \\
&= \frac{1}{c_n^2} \int_{\mathcal{R}} G^2\, d\boldsymbol{\xi} - \sum_{i=1}^{m} \frac{\phi_i(\boldsymbol{x})^2}{\lambda_i^2} \\
\sum_{i=1}^{m} \frac{\phi_i(\boldsymbol{x})^2}{\lambda_i^2} &\leq \frac{1}{c_n^2} \int_{\mathcal{R}} G^2\, d\boldsymbol{\xi} \\
\sum_{i=1}^{m} \int_{\mathcal{R}} \frac{\phi_i(\boldsymbol{x})^2}{\lambda_i^2}\, d\boldsymbol{x} &\leq \frac{1}{c_n^2} \int_{\mathcal{R}} \int_{\mathcal{R}} G^2\, d\boldsymbol{\xi}\, d\boldsymbol{x} \\
\sum_{i=1}^{m} \frac{1}{\lambda_i^2} &\leq B^2 \text{ say}
\end{aligned}$$

From this it follows that if λ is an m-fold repeated eigenvalue; i.e. $\lambda_1 = \cdots = \lambda_m = \lambda$, then

$$\frac{m}{\lambda_2} \leq B^2 \ ; \ m \leq \lambda^2 B^2$$

so that no eigenvalue can have more than finite multiplicity.

Similarly, if $(0 <)\lambda_i < M$ for $i = 1, \ldots, m$, then $m \leq M^2 B^2$, so that there are only a finite number of eigenvalues in any finite interval $[0, M]$.

We say that the eigenvalues form a **discrete spectrum**.

For example, for $\mathcal{R} = [0, \pi] \times [0, \pi]$; that is a square of side π in $\mathbb{R}^2$, the eigenfunctions are $\phi_{ij} = \frac{2}{\pi}\sin(ix)\sin(jy)$ and the eigenvalues are $\lambda_{ij} = i^2 + j^2$.

As in the one-dimensional case considered earlier, the fundamental eigenfunction, corresponding to the smallest eigenvalue is didtinguished by the fact that it does not change sign on $\mathcal{R}$.

We can approximate the smallest eigenvalue and the corresponding eigenfunction by using what is called the Rayleigh quotient.

Suppose that u is some sufficiently well behaved function in $\mathcal{R}$, which vanishes on $\partial\mathcal{R}$.

If we represent u as $u = \sum_{i=1}^{\infty} a_i \phi_i$, then

$$\begin{aligned}
\int_{\mathcal{R}} u^2\, dV &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i a_j \int_{\mathcal{R}} \phi_i \phi_j\, dV \\
&= \sum_{i=1}^{\infty} a_i^2 \\
\int_{\mathcal{R}} (\nabla u)^2\, dV &= -\int_{\mathcal{R}} u \nabla^2 u\, dv + \int_{\partial\mathcal{R}} u \frac{\partial u}{\partial n}\, dS \\
&= \int_{\mathcal{R}} \left(\sum_{j=1}^{\infty} a_j \phi_j\right)\left(\sum_{i=1}^{\infty} \lambda_i a_i \phi_i\right) dV \\
&= \sum_{i=1}^{\infty} \lambda_i a_i^2 \\
&\geq \lambda_1 \sum_{i=1}^{\infty} a_i^2 = \lambda_1 \int_{\mathcal{R}} u^2\, dV
\end{aligned}$$

The ratio

$$\rho(u) = \int_{\mathcal{R}} (\nabla u)^2\, dV \Bigg/ \int_{\mathcal{R}} u^2\, dV$$

is called the Rayleigh quotient. We have $\rho(u) \geq \lambda_1$, and the minimum is attained when $u = \phi_1$. Therefore, in much the same way as we found approximate solutions for the Dirichlet problem we can find the approximate value of the smallest eigenvalue and the corresponding eigenfunction by minimizing the Rayleigh quotient over a suitable class of functions.

For example, for the circular drum $r \leq 1$ in $\mathbb{R}^2$, we can consider the set of functions $u = 1 - r^a$.

We have

$$
\begin{aligned}
u &= 1 - r^a \\
u' &= -ar^{a-1} \\
u'' &= a(a-1)r^{a-2} \\
\nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = a^2 r^{a-2} \\
\iint_{\mathcal{R}} (\nabla u)^2 \, dS &= -\iint_{\mathcal{R}} u \nabla^2 u \, dS \\
&= a^2 \int_0^{2\pi} d\theta \int_0^1 (r^{a-2} - r^{2a-2}) r \, dr \\
&= (2\pi a^2) \left[\frac{1}{a} r^a - \frac{1}{2a} r^{2a} \right]_0^1 \\
&= \pi a \\
\iint_{\mathcal{R}} u^2 \, dS &= \int_0^{2\pi} \int_0^1 (1 - 2r^a + r^{2a}) r \, dr \\
&= 2\pi \left[\frac{1}{2} r^2 - \frac{2}{a+2} r^{a+2} + \frac{1}{2a+2} r^{2a+2} \right]_0^1 \\
&= \pi \left(1 - \frac{4}{a+2} + \frac{1}{a+1} \right) = \frac{\pi a^2}{a^2 + 3a + 2} \\
\rho(u) &= a + 3 + \frac{2}{a} \\
\frac{d\rho}{da} &= 1 - \frac{2}{a^2} = 0 \text{ when } a = \sqrt{2} \\
\lambda_i &\sim 3 + 2\sqrt{2}
\end{aligned}
$$

This approximate value 5.828 for the eigenvalue compares reasonably with the exact value 5.783.