MATH 3403
WEEK 7

Green's Functions.

For an arbitrary point $Q = (\xi, \eta, \zeta)$ in $\mathbb{R}^3$, consider the problem of finding a solution of Laplace's equation which is independent of the orientation with respect to Q, so that it is a function of $r = ((x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2)^{1/2}$ alone.

In spherical polar co-ordinates Laplace's equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 \ .$$

If $u = u(r)$,

$$(r^2u')' = 0 \ ; \ r^2u' = a \ ; \ u = b - a/r \ .$$

The solution $1/r$ is called a *Fundamental solution* of Laplace's equation. The equivalent solution in two dimensions is $-\log r$.

Now consider the problem of solving the Dirichlet problem for Laplace's equation in a bounded region $\mathcal{R}$ in $\mathbb{R}^3$ with boundary $\partial\mathcal{R}$.

If $Q = (\xi, \eta, \zeta)$ is an interior point of $\mathcal{R}$, then the Fundamental solution $1/r$ is harmonic in $\mathcal{R} \setminus \{Q\}$.

Green's identity

$$\iiint_{\mathcal{R}} (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \iint_{\partial\mathcal{R}} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS$$

only applies if the functions ϕ and ψ are harmonic throughout $\mathcal{R}$, so we initially excise from $\mathcal{R}$ a sphere σ of radius ϵ centered at Q. Since Q is an interior point we can ensure that $\sigma \subset \mathcal{R}$. Now, if ϕ is any function harmonic in $\mathcal{R}$ and $\psi = 1/r$, the integrand on the left of the identity vanishes on $\mathcal{R} \setminus \sigma$, so that

$$0 = \iint_{\partial\mathcal{R}} \left(\phi\frac{\partial(1/r)}{\partial n} - \frac{1}{r}\frac{\partial\phi}{\partial n}\right) dS + \iint_{\partial\sigma} \left(\phi\frac{\partial(1/r)}{\partial n} - \frac{1}{r}\frac{\partial\phi}{\partial n}\right) dS$$

On $\partial\sigma$, $\partial/\partial n = -\partial/\partial r$, since the normal *outwards* from $\mathcal{R} \setminus \sigma$ is directed *into* the sphere. Hence

$$\iint_{\partial\mathcal{R}} \left(\phi\frac{\partial(1/r)}{\partial n} - \frac{1}{r}\frac{\partial\phi}{\partial n}\right) dS = \iint_{\partial\sigma} \left(\phi\frac{\partial(1/r)}{\partial r} - \frac{1}{r}\frac{\partial\phi}{\partial r}\right) dS$$

On $\partial\sigma$

$$\frac{\partial(1/r)}{\partial r} = -\frac{1}{r^2} = -\frac{1}{\epsilon^2} \ ; \ \frac{1}{r} = \frac{1}{\epsilon} \quad \text{and } dS = \epsilon^2 d\Omega$$

where Ω is the solid angle.

Therefore the integral on the right hand side reduces to

$$-\iint_{\Omega} \left(\phi + \epsilon\frac{\partial\phi}{\partial r}\right) d\Omega \ .$$

Taking the limit as $\epsilon \to 0$, gives

$$\iint_{\partial\mathcal{R}} \left(\phi\frac{\partial(1/r)}{\partial n} - \frac{1}{r}\frac{\partial\phi}{\partial n}\right) dS = -4\pi\phi(Q) \ ,$$

or

$$\phi(Q) = -\frac{1}{4\pi} \iint_{\partial\mathcal{R}} \left(\phi \frac{\partial(1/r)}{\partial n} - \frac{1}{r}\frac{\partial \phi}{\partial n} \right) dS \ .$$

This integral expresses the value of ϕ at an interior point in terms of the boundary values of the function and its normal derivative. In this form it is not particularly useful, since for the Dirichlet problem only the values of the function and not those of the normal derivative are specified, and furthermore these are sufficient to determine the function completely in $\mathcal{R}$.

However, suppose that h is any function harmonic on $\mathcal{R}$. Since h is harmonic in $\mathcal{R}$,

$$0 = \iint_{\partial\mathcal{R}} \left(\phi \frac{\partial h}{\partial n} - h \frac{\partial \phi}{\partial n} \right) dS$$

so that

$$\phi(Q) = -\frac{1}{4\pi} \iint_{\partial\mathcal{R}} \left(\phi \frac{\partial}{\partial n} \left(\frac{1}{r} + h \right) - \left(\frac{1}{r} + h \right) \frac{\partial \phi}{\partial n} \right) dS$$

If we now choose for h the solution of the Dirichlet problem on $\mathcal{R}$ which takes the values $-1/r$ on $\partial\mathcal{R}$, and set $G = h + 1/r$ we obtain

$$\phi(Q) = -\frac{1}{4\pi} \iint_{\partial\mathcal{R}} \phi \frac{\partial G}{\partial n} \, dS$$

since $G = h + 1/r$ vanishes on $\partial\mathcal{R}$.

This function G is called the **Green's function** for the region $\mathcal{R}$.

In two dimensions we replace $\psi = 1/r$ by $\psi = -\log r$, which leads to

$$\phi(\xi, \eta) = -\frac{1}{2\pi} \oint_{\partial\mathcal{R}} \phi \frac{\partial G}{\partial n} \, ds$$

where

$$G = h - \log r \ ; \ \nabla^2 h = 0 \ \text{ in } \ \mathcal{R} \ ; G = 0 \ \text{ on } \ \partial\mathcal{R} \ .$$

Since $1/r$ (and $-\log r$) $\to \infty$ as $r \to 0$, we can choose ϵ so that $G > 0$ on $\partial\sigma$. Since G is harmonic on $\mathcal{R} \setminus \sigma$, it takes its minimum value on the boundary. We know that $G = 0$ on $\partial\mathcal{R}$, and since it is positive on $\partial\sigma$ the minimum value of G is zero, and occurs on $\partial\mathcal{R}$. Hence $G > 0$ in $\mathcal{R}$, and $\partial G/\partial n < 0$ on $\partial\mathcal{R}$.

In order to find G we have to solve a Dirchlet problem in $\mathcal{R}$ with boundary data which involves the general point Q. As a result, the Green's Function is usually only found for domains which have some regularity property which can be exploited. However, once the Green's function has been found all other Dirichlet problems can be solved by a boundary integral.

The symmetry of the Green's Function.

From its construction, we see that the Green's Function is a function of both the normal co-ordinate variables (e.g. x, y and z in $\mathbb{R}^3$) and the co-ordinates of the point Q. This relationship can be reflected by writing these functions in the form $G(\boldsymbol{x}, \boldsymbol{\xi})$.

Consider now two versions of the Green's function G for a domain $\mathcal{R}$ referring to distinct points $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$.

If we excise from $\mathcal{R}$ non-intersecting spheres of radius ϵ, σ_1 about $\boldsymbol{\xi}$ and σ_2 about $\boldsymbol{\eta}$, the two functions $u = G(\boldsymbol{x}, \boldsymbol{\xi})$ and $v = G(\boldsymbol{x}, \boldsymbol{\eta})$ are both harmonic in $\mathcal{R} \backslash (\sigma_1 \cup \sigma_2)$, so that applying the Green's identity

$$\iiint_{\mathcal{D}} (u\nabla^2 v - v\nabla^2 u)\, dV = \iint_{\partial\mathcal{D}} (uv_n - vu_n)\, dS$$

we obtain

$$0 = \iint_{\partial\mathcal{R}} (uv_n - vu_n)\, dS + \iint_{\partial\sigma_1} (uv_n - vu_n)\, dS + \iint_{\partial\sigma_2} (uv_n - vu_n)\, dS\ .$$

Both u and v vanish on $\partial\mathcal{R}$, so that we have

$$\iint_{\partial\sigma_1} \left(v\frac{\partial G(\boldsymbol{x}, \boldsymbol{\xi})}{\partial n} - \frac{\partial v}{\partial n} G(\boldsymbol{x}, \boldsymbol{\xi}) \right) dS = \iint_{\partial\sigma_2} \left(u\frac{\partial G(\boldsymbol{x}, \boldsymbol{\eta})}{\partial n} - \frac{\partial u}{\partial n} G(\boldsymbol{x}, \boldsymbol{\eta}) \right) dS\ .$$

In the same way as with our previous calculations, the left-hand side of this equation reduces to $4\pi v(\boldsymbol{\xi})$ as $\epsilon \to 0$, while the right hand side reduces to $4\pi u(\boldsymbol{\eta})$. Hence we have shown that

$$G(\boldsymbol{\xi}, \boldsymbol{\eta}) = G(\boldsymbol{\eta}, \boldsymbol{\xi})$$

and the Green's Function is symmetric with respect to its two sets of variables.

The method of reflections.

For some simple regions, geometrical considerations can be used to find the function h. In particular, Green's functions can be found for circular and spherical domains and for half spaces.

For example, if we take for $\mathcal{R}$ the half space $z > 0$, and $Q = (\xi, \eta, \zeta),\ \zeta > 0$, then on the boundary $z = 0$, r takes the value $((x-\xi)^2 + (y-\eta)^2 + \zeta^2)^{1/2}$ which is the same as the boundary value of $\rho = ((x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2)^{1/2}$. However, $1/\rho$ is harmonic everywhere except at $(\xi, \eta, -\zeta)$ which is not in $\mathcal{R}$ if $\zeta > 0$. Therefore in this case

$$G = \frac{1}{r} - \frac{1}{\rho}\ .$$

$$\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial z} = \frac{z-\zeta}{r^3} - \frac{z+\zeta}{\rho^3}$$

$$= \frac{-2\zeta}{(x-\xi)^2 + (y-\eta)^2 + \zeta^2)^{3/2}} \qquad \text{on} \qquad z = 0$$

$$\phi(\xi, \eta, \zeta) = \frac{\zeta}{2\pi} \int_{-\infty}^{\infty} dx \int_{\infty}^{\infty} dy \left(\frac{\phi(x, y, 0)}{(x-\xi)^2 + (y-\eta)^2 + \zeta^2)^{3/2}} \right)$$

Since the point $(\xi, \eta, -\zeta)$ is the reflection in the boundary $z = 0$ of the point $Q = (\xi, \eta, \zeta)$, this method is known as the method of reflections.

Circular regions.

We have already derived Poisson's integral formula from the series form. It can also be found by using a variation of the method of reflections to find the Green's function appropriate to a circular region.

Suppose that $\mathcal{R}$ is the interior of the circle $x^2 + y^2 < R^2$ in the plane. It is convenient to use polar co-ordinates ρ and ϕ to describe the position

$$\xi = \rho\cos\phi \ , \qquad \eta = \rho\sin\phi$$

of the logarithmic singularity of $G = G(x, y; \xi, \eta)$. The point inverse to (ξ, η) in the circle has polar co-ordinates R^2/ρ and ϕ. If $(x, y) = (R\cos\theta, R\sin\theta)$ is a point on the circumference of the circle (i.e. on $\partial\mathcal{R}$), then, denoting by r and $\hat{r}$ the distances from (x, y) to (ξ, η) and its inverse, we have

$$\begin{aligned}
r^2 &= (R\cos\theta - \rho\cos\phi)^2 + (R\sin\theta - \rho\sin\phi)^2 \\
&= R^2(\cos^2\theta + \sin^2\theta) - 2R\rho(\cos\theta\cos\phi + \sin\theta\sin\phi) + \rho^2(\cos^2\phi + \sin^2\phi) \\
&= R^2 - 2R\rho\cos(\theta - \phi) + \rho^2 \\
\hat{r}^2 &= R^2 - 2R\left(\frac{R^2}{\rho}\right)\cos(\theta - \phi) + \left(\frac{R^2}{\rho}\right)^2 \\
&= \left(\frac{R^2}{\rho^2}\right) r^2
\end{aligned}$$

Therefore $\frac{\rho}{R}\hat{r}$ takes the same values on the circle as r, and

$$\log\left(\frac{\rho\hat{r}}{R}\right) - \log r = \log\left(\frac{\rho\hat{r}}{Rr}\right) = 0 \qquad \text{on} \qquad \partial\mathcal{R} \ .$$

Hence the Green's function for the circle (with $x = s\cos\theta$ and $y = s\sin\theta$) is

$$\begin{aligned}
G &= \frac{1}{2}\log\left(\frac{\rho^2\hat{r}^2}{R^2 r^2}\right) \\
&= \frac{1}{2}\Bigg(\log\left(\frac{\rho^2}{R^2}((x - (R^2/\rho)\cos\phi)^2 + (y - (R^2/\rho)\sin\phi)^2\right) \\
&\qquad - \log\big((x - \rho\cos\phi)^2 + (y - \rho\sin\phi)^2\big)\Bigg) \\
&= \frac{1}{2}\left(\log\left(\frac{\rho^2 s^2}{R^2} - 2\rho s\cos(\theta - \phi) + R^2\right) - \log(s^2 - 2\rho s\cos(\theta - \phi) + \rho^2)\right) \\
\frac{\partial G}{\partial n} &= \frac{\partial G}{\partial s} \\
&= \frac{1}{2}\left(\frac{\frac{2s\rho^2}{R^2} - 2\rho\cos(\theta - \phi)}{\frac{\rho^2 s^2}{R^2} - 2\rho s\cos(\theta - \phi) + R^2} - \frac{2s - 2\rho\cos(\theta - \phi)}{s^2 - 2\rho s\cos(\theta - \phi) + \rho^2}\right)
\end{aligned}$$

When $s = R$,

$$\begin{aligned}
\frac{\partial G}{\partial n} &= \frac{\frac{\rho^2}{R} - \rho\cos(\theta - \phi)}{\rho^2 - 2\rho R\cos(\theta - \phi) + R^2} - \frac{R - \rho\cos(\theta - \phi)}{R^2 - 2\rho R\cos(\theta - \phi) + \rho^2} \\
&= -\frac{1}{R}\frac{R^2 - \rho^2}{R^2 - 2R\rho\cos(\theta - \phi) + \rho^2}
\end{aligned}$$

which is in agreement with the earlier result.

The Poisson Equation.

The Green's function can also be used to find the solution of the *Poisson Equation*

$$\nabla^2 u = F \ \text{ in } \ \mathcal{R} \ ; \ u = 0 \ \text{ on } \ \partial\mathcal{R} \ .$$

Suppose that G is the Green's function for $\mathcal{R}$ with singularity at $Q = (\xi, \eta, \zeta)$. As before, let σ be a sphere, centre Q and radius ϵ lying in $\mathcal{R}$. We consider the Green's identity

$$\iiint_{\mathcal{R}\setminus\sigma} \left(u\nabla^2 G - G\nabla^2 u\right) \, dV = \iint_{\partial\mathcal{R}} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) \, dS + \iint_{\partial\sigma} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) \, dS \ .$$

In $\mathcal{R} \setminus \sigma$, $\nabla^2 G = 0$ and $\nabla^2 u = F$, while on $\partial\mathcal{R}$, $G = 0$ and $u = 0$. On $\partial\sigma$, $\partial/\partial n = -\partial/\partial r$, $r = \epsilon$, and $dS = \epsilon^2 d\Omega$.

Therefore

$$\begin{aligned}
-\iiint_{\mathcal{R}\setminus\sigma} G(\boldsymbol{x},\boldsymbol{\xi})F(\boldsymbol{x})\, dV &= -\iint_{\partial\sigma} \left(u\frac{\partial G}{\partial r} - G\frac{\partial u}{\partial r}\right) \, dS \\
&= \iint_{\partial\sigma} \left(u\left(\frac{1}{r^2} - h_r\right) + \left(\frac{1}{r} + h\right) u_r\right) \, dS \\
&= \iint_{\Omega} \left(u\left(\frac{1}{\epsilon^2} - h_r\right) + \left(\frac{1}{\epsilon} + h\right) u_r\right) \epsilon^2 \, d\Omega \\
&= \iint_{\Omega} \left(u + \epsilon u_r + \epsilon^2(h u_r - u h_r)\right) \, d\Omega
\end{aligned}$$

Taking the limit $\epsilon \to 0$, we obtain

$$\begin{aligned}
4\pi u(\boldsymbol{\xi}) &= -\iiint_{\mathcal{R}} G(\boldsymbol{x},\boldsymbol{\xi})F(\boldsymbol{x})\, dV \\
u(\boldsymbol{\xi}) &= -\frac{1}{4\pi} \iint_{\mathcal{R}} G(\boldsymbol{x},\boldsymbol{\xi})F(\boldsymbol{x})\, dV
\end{aligned}$$

or, equivalently,

$$u(\boldsymbol{x}) = -\frac{1}{4\pi} \iint_{\mathcal{R}} G(\boldsymbol{x},\boldsymbol{\xi})F(\boldsymbol{\xi})\, dV$$

In two dimensions the solution is modified appropriately to give

$$u(\boldsymbol{\xi}) = -\frac{1}{2\pi} \iint_{\mathcal{R}} G(\boldsymbol{x},\boldsymbol{\xi})F(\boldsymbol{x})\, dS \ ,$$

while in one dimension it becomes

$$u(\xi) = -\frac{1}{2} \int_{\mathcal{R}} G(x,\xi)F(x)\, dx \ .$$

If we have the more general Poisson equation

$$\nabla^2 u = F \ \text{ in } \ \mathcal{R} \ ; \ u = f \ \text{ on } \ \partial\mathcal{R}$$

we can construct the solution as the sum of the solution given above and the solution

$$\phi(\boldsymbol{\xi}) = -\frac{1}{4\pi} \iint_{\partial\mathcal{R}} f\frac{\partial G}{\partial n}\, dS$$

of the Dirichlet problem

$$\nabla^2 \phi = 0 \ \text{ in } \ \mathcal{R} \ ; \ \phi = f \ \text{ on } \ \partial\mathcal{R} \ .$$

Green's Functions and eigenfunctions.

Consider the problem of finding the Green's function for Laplace's equation in one dimension for the interval $[0,1]$.

We are looking for a function $G(x,\xi)$ on $(0,1)\times(0,1)$ which satisfies

$$\frac{d^2G}{dx^2} = 0 \quad \text{on} \quad (0,1)\setminus\{\xi\}\ ; \quad G(0,\xi) = G(1,\xi) = 0\ .$$

We have seen that the functions $\sin n\pi x$, $n = 1, 2, \ldots$ form a complete set of eigenfunctions which satisfy the boundary conditions on this interval. Suppose therefore that we expand $G(x,\xi)$ as a series

$$G(x,\xi) = \sum_{n=1}^{\infty} a_n(\xi) \sin n\pi x\ .$$

The coefficients $a_n(\xi)$ will be given by

$$\begin{aligned} a_n(\xi) &= \int_0^1 G(x,\xi)\sin n\pi x\,dx \Big/ \int_0^1 \sin^2 n\pi x\,dx \\ &= 2\int_0^1 G(x,\xi)\sin n\pi x\,dx \end{aligned}$$

The eigenfunctions $\phi_n(x) = \sin n\pi x$ satisfy

$$\frac{d^2\phi_n}{dx^2} = -n^2\pi^2 \sin n\pi x \quad (= -\lambda_n\phi_n)$$

which is Poisson's equation in one dimension.

Therefore we can use the one-dimensional form of the previous result to give

$$\begin{aligned} \sin n\pi\xi &= -\frac{1}{2}\int_0^1 G(x,\xi)(-n^2\pi^2\sin n\pi x)\,dx \\ 2\int_0^1 G(x,\xi)\sin n\pi x\,dx &= \frac{4}{n^2\pi^2}\sin n\pi\xi \\ G(x,\xi) &= 4\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\sin n\pi\xi\sin n\pi x \\ &= 2\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}(\sqrt{2}\sin n\pi\xi)(\sqrt{2}\sin n\pi x)\ . \end{aligned}$$

This is a particular case of the general result that the Green's Function can be expressed in the form

$$G(\boldsymbol{x},\boldsymbol{\xi}) = c_n\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\phi_n(\boldsymbol{x})\phi_n(\boldsymbol{\xi})\ ,$$

where the functions ϕ_n are normalised (i.e. $\int \phi_n^2 = 1$) eigenfunctions satisfying

$$\nabla^2\phi_n + \lambda_n\phi_n = 0 \quad \text{in} \quad \mathcal{R}\ ; \ \phi_n = 0 \quad \text{on} \quad \partial\mathcal{R}\ ,$$

and c_n is the scale factor $(2,\ 2\pi,\ 4\pi)$ depending on the dimension.

Some authors combine this scale factor into the Green's function.