MATH 3403

WEEK 5

The Finite string.

When we have the wave equation on a finite x domain, the approach which we have used so far is of limited usefulness.

Suppose that we wish to solve

$$u_{xx} = c^{-2} u_{tt}\ ,\ 0 < t < \infty\ ,\ 0 < x < l$$

Firstly: the D'Alembert solution is only defined in the small triangle bounded by the x-axis and the characteristics $x - ct = 0$ and $x + ct = l$.

Secondly: both the lines $x = 0$ and $x = l$ are *timelike*, so that we need to specify one piece of boundary data on each of them.

If we specify homogeneous fixed or free data (i.e. $u = 0$ or $u_x = 0$) at the end points, we can (in theory at least) use the reflection property to extend the initial data onto the whole x- axis, and then use the full-range D'Alembert solution.

For example, suppose we have to solve

$$\begin{aligned} u_{xx} &= u_{tt}\ ,\ 0 < t < \infty\ ,\ 0 < x < 1 \\ u(x,0) &= x(1-x)\ ;\ u_t(x,0) = 0\ ;\ 0 < x < 1 \\ u(0,t) &= u(1,t) = 0\ ;\ 0 < t < \infty \end{aligned}$$

Because we have fixed endpoints, we obtain an equivalent solution by considering the odd extensions of the initial data.

In the case of u_t this is simple; $u_t(x,0) = 0\ ,\ -\infty < x < \infty$.

For $u(x,0)$, we have

$$\begin{aligned} u(x,0) &= (x+1)x \quad -1 < x < 0 \\ &= (x-1)(x-2) \quad 1 < x < 2 \\ &= -(x+2)(x+1) \quad -2 < x < -1 \\ &= -(x-2)(x-3) \quad 2 < x < 3 \\ &= (-1)^n (x-(n-1))(x-n) \quad n-1 < x < n \end{aligned}$$

Hence the solution at (x_0, t_0) is $\frac{1}{2}(f(x_0 - t_0) + f(x_0 + t_0))$ for this initial function $f(x) = u(x,0)$.

If the integral part of $x_0 - t_0$ is n_1, so that $x_0 - t_0 = n_1 + r_1$, $0 \le r_1 < 1$, and similarly $x_0 + t_0 = n_2 + r_2$, then

$$u(x_0, t_0) = \frac{1}{2}\left((-1)^{n_1} r_1(1-r_1) + (-1)^{n_2} r_2(1-r_2)\right)\ .$$

While this procedure can be used to evaluate the solution for particular values, it's usefulness is limited to these specific boundary conditions.

Instead, we will seek to express the solution as an infinite series of functions. Since we cannot evaluate infinite sums in general, this amounts to determining a sequence of (one hopes) increasingly accurate approximations.

Eigenvalues and Eigenfunctions.

Consider the differential equation

$$y'' + \lambda y = 0 \ .$$

If we impose general *homogeneous boundary conditions* of the form

$$\begin{aligned} k_{11}y(a) + k_{12}y'(a) &= 0 \qquad (k_{11}, k_{12}) \neq (0,0) \\ k_{21}y(b) + k_{22}y'(b) &= 0 \qquad (k_{21}, k_{22}) \neq (0,0) \end{aligned}$$

we can ensure that there will always be at least one solution to the boundary value problem, namely $y \equiv 0$. We can now ask the question whether there are values of λ for which an infinite set of solutions exist. Such values of λ are called *eigenvalues* and the corresponding solutions are called *eigenfunctions*.

e.g. 1.

Consider the problem

$$y'' + \lambda y = 0 \ ; \ y(0) = 0 \ , \ y(1) = 0 \ .$$

If $\lambda > 0$, set $\lambda = \omega^2$. Then the general solution of the equation

$$y'' + \omega^2 y = 0$$

is

$$y = a\cos(\omega x) + b\sin(\omega x) \ .$$

The condition $y(0) = 0$ implies $a = 0$, and now the second condition gives

$$b\sin(\omega) = 0 \ .$$

As well as the solution $b = 0$ which leads to $y \equiv 0$, we have the possibility $\sin(\omega) = 0$, which has the solutions $\omega = n\pi$. Therefore, if $\lambda = n^2\pi^2$, $n = 1, 2, \ldots$,we have the non-trivial solution sets $y = b\sin(n\pi x)$.

If $\lambda < 0$, set $\lambda = -p^2$. Then the general solution of the equation

$$y'' - p^2 y = 0$$

can be written

$$y = a\cosh(px) + b\sinh(px) \ .$$

The condition $y(0) = 0$ implies $a = 0$ again, and the second condition gives

$$b\sinh(p) = 0 \ .$$

However, the function $\sinh x$ vanishes only for $x = 0$, and since $p \neq 0$ we must take $b = 0$ and hence $y \equiv 0$.

Finally, if $\lambda = 0$, the general solution of

$$y'' = 0$$

is

$$y = a + bx \ ,$$

and the boundary conditions ensure that $a = b = 0$ and $y \equiv 0$.

Therefore the eigenvalues for this problem are $\lambda = n^2\pi^2$ and the corresponding eigenfunctions are $y(x) = \sin(n\pi x)$.

e.g. 2.

Consider the boundary value problem

$$y'' + \lambda y = 0 \; ; y'(0) = 0 \; , \; y'(1) = 0 \; .$$

For $\lambda < 0$ set $\lambda = -p^2$. Then the general solution of

$$y'' - p^2 y = 0$$

can be written in the form

$$y = a\cosh(px) + b\sinh(px) \; ; \; y' = pa\sinh(px) + pb\cosh(px)$$

The condition $y'(0) = 0$ gives $b = 0$, and the condition $y'(1) = 0$ gives

$$pa\sinh(p) = 0 \; ; \; a = 0 \quad \text{since} \quad p \neq 0 \; .$$

For $\lambda = 0$, the general solution is

$$y = a + bx \; ; \; y' = b$$

so that the conditions give $b = 0$, and $y = a$ is an infinite solution set. That is, $\lambda = 0$ is an eigenvalue for this problem, and the corresponding eigenfunction can be taken as $y \equiv 1$.

For $\lambda > 0$ set $\lambda = \omega^2$. The general solution of

$$y'' + \omega^2 y = 0$$

is

$$y = a\cos(\omega x) + b\sin(\omega x) \; ; \; y' = -a\omega\sin(\omega x) + b\omega\cos(\omega x) \; .$$

The condition $y'(0) = 0$ gives $b = 0$, and the condition $y'(1) = 0$ gives

$$a\omega\sin(\omega) = 0$$

so that, as well as the solution $a = 0$ we have $\omega = n\pi$, leading to the eigenvalues $\lambda = n^2\pi^2$ and the corresponding eigenfunctions $y = \cos(n\pi x)$.

e.g. 3.

Consider the problem

$$y'' + \lambda y = 0 \; ; \; y(0) = 0 \; , \; y'(1) = 2y(1) \; .$$

For $\lambda < 0$, set $\lambda = -p^2$. The general solution of

$$y'' - p^2 y = 0$$

is

$$y = a\cosh(px) + b\sinh(px) \; ,$$

and the condition $y(0) = 0$ gives $a = 0$. The second condition gives

$$\begin{aligned} pb\cosh(p) &= 2b\sinh(p) \\ \tanh(p) &= \frac{p}{2} \quad \text{if } b \neq 0 \\ p &\simeq 1.915 \end{aligned}$$

Therefore one eigenvalue is $-(1.915)^2$, and the corresponding eigenfunction is $y = \sinh(1.915x)$.

For $\lambda = 0$, the general solution is $y = a + bx$. The condition $y(0) = 0$ gives $a = 0$, and the condition $y'(1) = 2y(1)$ gives $b = 2b$, $b = 0$, so that the only solution is $y \equiv 0$.

Finally, for $\lambda > 0$, we set $\lambda = \omega^2$, so that the general solution is

$$y = a\cos(\omega x) + b\sin(\omega x) \ .$$

The condition $y(0) = 0$ gives $a = 0$, and the condition $y'(1) = 2y(1)$ gives

$$\begin{aligned} b\omega\cos(\omega) &= 2b\sin(\omega) \\ \tan(\omega) &= \frac{\omega}{2} \quad \text{if } b \neq 0 \end{aligned}$$

This equation has a sequence of solutions

$$\omega_n \simeq \left(n + \frac{1}{2}\right)\pi - \frac{4}{(2n+1)\pi} \qquad n = 1, 2, \ldots$$

so that the eigenvalues are $\lambda = \omega_n^2$, and the corresponding eigenfunctions are $y = \sin(\omega_n x)$.

A common feature of these examples, which is characteristic of all such eigenvalue problems on finite intervals, is that the eigenvalues are discrete and bounded below.

Furthermore, suppose that y is a non-trivial solution of

$$y'' + \lambda y = 0$$

on $[a, b]$ with appropriate boundary conditions.

Then

$$\begin{aligned} \int_a^b yy''\,dx + \lambda \int_a^b y^2\,dx = 0 \\ yy'|_a^b - \int_a^b (y')^2\,dx + \lambda \int_a^b y^2\,dx = 0 \\ \lambda \int_a^b y^2\,dx = \int_a^b (y')^2\,dx - y(b)y'(b) + y(a)y'(a) \end{aligned}$$

If the boundary condition at b is either $y(b) = 0$ or $y'(b) = 0$ and the condition at a is also either $y(a) = 0$ or $y'(a) = 0$, then this reduces to

$$\lambda = \int_a^b (y')^2\,dx \Big/ \int_a^b y^2\,dx \geq 0$$

and to have $\lambda = 0$ we require $y' = 0$ throughout $[a, b]$, so that $y'(a) = y'(b) = 0$.

In this case, as we have seen, $y = 1$ is an eigenfunction corresponding to $\lambda = 0$. All the other eigenvalues in these cases are positive, and the corresponding eigenfunctions are trigonometric.

Where we have boundary conditions of the third kind, the eigenvalues will still all be positive provided

$$
\begin{aligned}
y(b)y'(b) &\le 0 \\
y(a)y'(a) &\ge 0
\end{aligned}
$$

This means that at a we have

$$y'(a) = ky(a) \ ; \ k > 0$$

while at b we have

$$y'(b) + ky(b) = 0 \ ; \ k > 0$$

Negative eigenvalues can only possibly occur when there are boundary conditions of the form

$$y'(a) + ky(a) = 0 \ ; \ k > 0$$

and/or

$$y'(b) = ky(b) \ ; \ k > 0 \ .$$

Note also that the eigenfunction corresponding to the smallest eigenvalue does not cross the axis in the open interval $(0, 1)$, the next eigenfunction crosses the axis once in this interval, the third one twice and so on. This property provides a useful check that no eigenfunctions have been overlooked.

Orthogonality.

The final important property of the eigenfunctions is their orthogonality.

Suppose that we have the general boundary value problem

$$y'' + \lambda y = 0 \ ; \ k_{11}y(a) + k_{12}y'(a) = 0 \ , \ k_{21}y(b) + k_{22}y'(b) = 0$$

This problem includes the examples above as special cases.

If y_m is an eigenfunction corresponding to the eigenvalue λ_m,
and y_n is an eigenfunction corresponding to the eigenvalue λ_n, where $\lambda_m \neq \lambda_n$, then

$$
\begin{aligned}
\lambda_m \int_a^b y_m y_n \, dx &= -\int_a^b y_m'' y_n \, dx \\
&= -y_m' y_n \big|_a^b + \int_a^b y_m' y_n' \, dx \\
&= -\, y_m'(b) y_n(b) + y_m'(a) y_n(a) + \int_a^b y_m' y_n' \, dx \\
\lambda_n \int_a^b y_m y_n \, dx &= -\, y_n'(b) y_m(b) + y_n'(a) y_m(a) + \int_a^b y_m' y_n' \, dx \\
(\lambda_m - \lambda_n) \int_a^b y_m y_n \, dx &= \begin{vmatrix} y_m(b) & y_m'(b) \\ y_n(b) & y_n'(b) \end{vmatrix} - \begin{vmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{vmatrix} \ .
\end{aligned}
$$

Since the equations

$$\begin{pmatrix} y_m(b) & y_m'(b) \\ y_n(b) & y_n'(b) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

have the non-trivial solution $(c_1, c_2)' = (k_{21}, k_{22})'$, the system has rank < 2, and hence the determinant of the coefficient matrix vanishes.

Similarly,

$$\begin{vmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{vmatrix} = 0 \ .$$

Since $\lambda_m \neq \lambda_n$, this gives

$$\int_a^b y_m y_n \, dx = 0 \ .$$

Hence the eigenfunctions are orthogonal on the interval $[a, b]$.

This means that, for finite sums at least, if

$$f(x) = \sum_{i=1}^{N} \alpha_i y_i(x)$$

then

$$\begin{aligned} \int_a^b f(x) y_k(x) \, dx &= \sum_{i=1}^{N} \alpha_i \int_a^b y_i(x) y_k(x) \, dx \\ &= \alpha_k \int_a^b y_k^2 \, dx \\ \alpha_k &= \int_a^b f y_k \, dx \Big/ \int_a^b y_k^2 \, dx \ . \end{aligned}$$

We assume that this approach extends to infinite sums as well.

The coefficients α_k are called *Fourier coefficients*. The orthogonality of the eigenfunctions means that they can be calculated independently of each other. Furthermore, changing the number of terms taken in any approximate expansion does not alter the coefficients.

In contrast, if we consider an expansion

$$f(x) = \sum_{k=1}^{N} \beta_k \psi_k(x)$$

where the functions $\psi_k(x)$ are not orthogonal, then calculating the coefficients β_k involves solving an $N \times N$ system of simultaneous equations

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} & \\ a_{21} & a_{22} & \dots & a_{2N} & \\ . & . & . & \dots & . \\ a_{N1} & a_{N2} & \dots & a_{NN} & \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ . \\ \beta_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ . \\ f_N \end{pmatrix}$$

where

$$\begin{aligned} a_{ij} &= \int_a^b \psi_i(x) \psi_j(x) \, dx \\ f_i &= \int_a^b \psi_i(x) f(x) \, dx \end{aligned}$$

and where the values β_k of the solution change whenever the value N changes.

THE METHOD OF SEPARATION OF VARIABLES

Consider the following case of the wave equation on a finite interval.

$$\begin{gathered}\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\\ u(0,t) = 0 \;;\; u(1,t) = 0\\ u(x,0) = f(x) \;;\; u_t(x,0) = g(x)\end{gathered}$$

The equation is linear, and we have homogeneous boundary conditions ($u(0,t) = 0$, $u(1,t) = 0$,for fixed values of the variable x. Therefore we can attempt to solve the equation by using the **method of separation of variables**.

This method consists firstly of trying to find non-trivial solutions of the partial differential equation

$$u_{xx} = c^{-2} u_{tt}$$

of the form $u = X(x)T(t)$ which satisfy the boundary conditions $u(0,t) = 0$, $u(1,t) = 0$, in x without worrying about the initial conditions in t. Substituting this form into the partial differential equation, we have

$$\begin{gathered}X''(x)T(t) = \frac{1}{c^2}X(x)\ddot{T}(t)\\ \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)}\end{gathered}$$

since we are assuming that $X(x)$ and $T(t)$ are not identically 0;

$$\frac{X''(x)}{X(x)} = -\lambda = \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)}$$

since the left hand side is independent of t and the right hand side is independent of x.

In particular,

$$X''(x) + \lambda X(x) = 0$$

for some constant λ, and the boundary conditions on u imply

$$X(0) = 0 \;,\; X(1) = 0 \;.$$

This is an eigenvalue problem for the function X.

As we have seen, the only non-trivial solutions of this problem correspond to $\lambda = n^2\pi^2$, $X(x) = \sin(n\pi x)$.

For these values of λ, $T(t)$ satisfies

$$\begin{gathered}\ddot{T}(t) + c^2 n^2 \pi^2 T(t) = 0\\ T(t) = a_n \cos(cn\pi t) + b_n \sin(cn\pi t)\end{gathered}$$

where a_n and b_n are arbitrary constants.

Note that it is the boundary conditions on the functions $X(x)$ which determine the eigenvalues, and which in turn determine the solutions $T(t)$.

Therefore the functions $(a_n \cos(cn\pi t) + b_n \sin(cn\pi t)) \sin(n\pi x)$ are solutions of

$$\begin{gathered}\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \\ u(0,t) = 0 \ , \ \ u(1,t) = 0\end{gathered}$$

for all choices of a_n and b_n, and for all $n = 1, 2, \dots$, and because the equation is linear, this is also true for arbitrary linear combinations of these solutions. By extension we assume that, subject to convergence, the infinite sum

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(cn\pi t) + b_n \sin(cn\pi t)) \sin(n\pi x)$$

is also a solution of this problem, and we will be able to solve the original problem provided we can choose the constants a_n and b_n so that $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Substituting into the expression for $u(x,t)$ this implies

$$\sum_{n=1}^{\infty} a_n \sin(n\pi x) = f(x) \ ,$$

and

$$\sum_{n=1}^{\infty} cn\pi b_n \sin(n\pi x) = g(x) \ .$$

Now using the orthogonality of the eigenvalues we see that the coefficients are given by Fourier's formula

$$a_k = \int_a^b f\phi_k \, dx \Bigg/ \int_a^b \phi_k^2 \, dx \ ,$$

so that

$$\begin{aligned} a_n &= \int_0^1 f(x)\sin(n\pi x)\, dx \Bigg/ \int_0^1 \sin^2(n\pi x)\, dx \\ &= 2\int_0^1 f(x)\sin(n\pi x)\, dx \\ b_n &= \frac{2}{cn\pi}\int_0^1 g(x)\sin(n\pi x)\, dx \ . \end{aligned}$$

For the purposes of this subject, we will assume that all these procedures are valid, but students should note that in order to justify this solution we need to ensure the convergence of the infinite sums which occur, prove that these sums are in fact twice differentiable with respect to both of the variables, and prove that it is possible to interchange the operations of differentiation and infinite summation.

In this particular case, the solution is a periodic function of t.

$$\begin{aligned} & u\left(x, t+\frac{2}{c}\right) \\ = & \sum_{n=1}^{\infty}\left(a_n \cos(cn\pi t + 2n\pi) + b_n \sin(cn\pi t + 2n\pi)\right)\sin(n\pi x) \\ & = u(x,t) \end{aligned}$$

This is a consequence of the regularity of the frequencies

$$\omega_n = n\pi \ .$$

This property is called *consonance*.

Where we have boundary conditions of the type $X'(a) = \alpha X(a)$, the frequencies are not regular, the solutions are not periodic, and we have *dissonnance*. This also occurs in vibrations of a circular membrane, where the frequencies are the zeros of Bessel Functions.

As a particular example, let us take $f(x) = x(1-x)$, and $g(x) = 0$. The b_n are all 0, while the a_n are given by

$$\begin{aligned} a_n &= 2\int_0^1 x(1-x)\sin(n\pi x)\,dx \\ &= -\frac{2}{n\pi}\cos(n\pi x)x(1-x)\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \cos(n\pi x)(1-2x)\,dx \\ &= \frac{2}{n^2\pi^2}\sin(n\pi x)(1-2x)\Big|_0^1 + \frac{4}{n^2\pi^2}\int_0^1 \sin(n\pi x)\,dx \\ &= -\frac{4}{n^3\pi^3}\cos(n\pi x)\Big|_0^1 \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8}{n^3\pi^3} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

The solution for $u(x,t)$ is

$$u(x,t) = \frac{8}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\cos(c(2k+1)\pi t)\sin((2k+1)\pi x) \ .$$

This provides an *a posteriori* justification for the solution process.

Since sin and cos are bounded functions,

$$\left|\frac{1}{(2k+1)^3}\cos(c(2k+1)\pi t)\sin((2k+1)\pi x)\right| \le \frac{1}{(2k+1)^3}$$

for all x and t.

Furthermore,

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}$$

converges, so that we have dominated convergence of $u(x,t)$, and therefore the convergence is absolute and uniform for all x and t.

The Finite Heat Equation.

The method of separation of variables is also applicable to the heat equation on finite space intervals and to Laplace's equation in finite domains.

Consider firstly the following heat equation problem.

$$\begin{gathered}\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 1 \ ; \ t > 0 \\ u(x,0) = 1 \ ; \ u(0,t) = 0 \ ; \ u_x(1,t) = 0\end{gathered}$$

This represents a body, initially at a uniform temperature 1, which has one end ($x = 0$) reduced (discontinuously) to temperature 0 at time $t = 0$, while the other end is insulated.

Since we have homogeneous boundary conditions at $x = 0$ and $x = 1$, we can apply the method of separation of variables.

Consider the boundary value problem

$$\begin{gathered}\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 1 \ ; \ t > 0 \\ u(0,t) = 0 \ ; \ u_x(1,t) = 0\end{gathered}$$

without reference to the initial condition.

The function $(x,t) = X(x)T(t)$ will be a solution of this problem if

$$\begin{gathered}X(x)\dot{T}(t) = \kappa X''(x)T(t) \\ \frac{1}{\kappa}\frac{\dot{T}}{T} = \frac{X''}{X} = -\lambda \\ X'' + \lambda X = 0 \ ; \ X(0) = 0 \ ; \ X'(1) = 0\end{gathered}$$

To solve this eigenvalue problem consider $\lambda = \omega^2$.

The general solution of the equation for X is

$$X = a \sin \omega x + b \cos \omega x$$

and applying the boundary conditions give

$$\begin{gathered}X(0) = b = 0 \\ X' = a\omega \cos \omega x \\ X'(1) = a\omega \cos \omega = 0 \\ \cos \omega = 0 \ ; \ \omega = \left(n + \frac{1}{2}\right)\pi \\ \lambda = \left(n + \frac{1}{2}\right)^2 \pi^2\end{gathered}$$

The eigenfunctions are

$$\sin\left(\left(n + \frac{1}{2}\right)\pi x\right) \ .$$

The function T satisfies

$$\begin{aligned}\dot{T} &= -\left(n+\frac{1}{2}\right)^2 \pi^2 \kappa T\\ T &= a_n \exp\left(-\left(n+\frac{1}{2}\right)^2 \pi^2 \kappa t\right)\end{aligned}$$

The general solution of the boundary value problem is

$$u(x,t) = \sum_{n=0}^{\infty} a_n \sin\left(\left(n+\frac{1}{2}\right)\pi x\right) \exp\left(-\left(n+\frac{1}{2}\right)^2 \pi^2 \kappa t\right)$$

Substituting for the initial conditions we have

$$1 = \sum_{n=0}^{\infty} a_n \sin\left(\left(n+\frac{1}{2}\right)\pi x\right)$$

$$\begin{aligned}&\int_0^1 \sin\left(\left(n+\frac{1}{2}\right)\pi x\right)\,dx\\ &= a_n \int_0^1 \sin^2\left(\left(n+\frac{1}{2}\right)\pi x\right)\,dx\end{aligned}$$

$$\begin{aligned}&-\frac{2}{(2n+1)\pi}\cos\left(\left(n+\frac{1}{2}\right)\pi x\right)\Big|_0^1\\ &= a_n \int_0^1 \left(\frac{1}{2}-\frac{1}{2}\cos((2n+1)\pi x)\right)\,dx\end{aligned}$$

$$\begin{aligned}\frac{2}{(2n+1)\pi} &= \frac{1}{2}a_n\\ a_n &= \frac{4}{(2n+1)\pi}\end{aligned}$$

$$u(x,t) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left(\left(n+\frac{1}{2}\right)\pi x\right) \exp\left(-\left(n+\frac{1}{2}\right)^2 \pi^2 \kappa t\right)$$

For $t \geq \tau > 0$, this series is dominated by the convergent series

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} \exp\left(-\left(n+\frac{1}{2}\right)^2 \pi^2 \kappa \tau\right)$$

so that the convergence is absolute and uniform, and the solution is continuous.

On the other hand, when $t = 0$, the solution is discontinuous, and the series does not converge uniformly. Instead, it displays the *Gibbs phenomenon*.

In this respect the heat equation differs from the wave equation.

For the wave equation, discontinuities in the initial data persist unaltered for all future time.

For the heat equation, discontinuities vanish instantly.

Separation of variables for elliptic problems.

As we shall see later, elliptic equations are typically solved on bounded domains, with one piece of boundary data specified at each point of the boundary. Where the boundary consists of segments along which the co-ordinates are constant, we can apply the method of separation of variables to obtain a solution.

For example, consider the following boundary value problem for *Laplace's Equation* in rectangular co-ordinates.

$$
\begin{gathered}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\
u(0,y) = 0 \ , \ u(\pi,y) = \frac{\pi}{2} - \left| \frac{\pi}{2} - y \right| \\
u(x,0) = 0 \ , \ u(x,\pi) = x^2(\pi - x) \ .
\end{gathered}
$$

While the boundary data are given along the lines $x = const$ and $y = const$, in neither case are the boundary conditions homogeneous, so that the method of separation of variables cannot be applied directly.

However, the differential equation is linear, so that we can determine the required solution as the sum of the solutions of the simpler problems

$$
\begin{gathered}
\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0 \\
u_1(0,y) = 0 \ , \ u_1(\pi,y) = \frac{\pi}{2} - \left| \frac{\pi}{2} - y \right| \\
u_1(x,0) = 0 \ , \ u_1(x,\pi) = 0 \ ,
\end{gathered}
$$

and

$$
\begin{gathered}
\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0 \\
u_2(0,y) = 0 \ , \ u_2(\pi,y) = 0 \\
u_2(x,0) = 0 \ , \ u_2(x,\pi) = x^2(\pi - x) \ .
\end{gathered}
$$

The first of these has homogeneous boundary data along the lines $y = 0$ and $y = \pi$, while the second has homogeneous boundary data along the lines $x = 0$ and $x = \pi$.

In solving the first part, we look for solutions $u_1 = X(x)Y(y)$, where $Y(0) = Y(\pi) = 0$. Making the substitution in the equation, we have

$$
\begin{gathered}
X''(x)Y(y) + X(x)Y''(y) = 0 \\
\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda \\
Y''(y) + \lambda Y(y) = 0 \ ; \ Y(0) = 0 \ , \ Y(\pi) = 0 \\
\lambda = n^2 \quad n = 1, 2, \ldots \qquad Y(y) = \sin(ny)
\end{gathered}
$$

If $\lambda = n^2$, then the functions $X(x)$ satisfy

$$\begin{aligned} X''(x) - n^2 X(x) &= 0 \\ X(x) &= a_n e^{nx} + b_n e^{-nx} \ , \end{aligned}$$

and the solution u_1 can be expressed as

$$u_1(x,y) = \sum_{n=1}^{\infty} \left(a_n e^{nx} + b_n e^{-nx} \right) \sin(ny) \ .$$

In order to satisfy the other boundary conditions, we need

$$\sum_{n=1}^{\infty} (a_n + b_n) \sin(ny) = 0$$

and

$$\sum_{n=1}^{\infty} \left(a_n e^{n\pi} + b_n e^{-n\pi} \right) \sin(ny) = \frac{\pi}{2} - \left| \frac{\pi}{2} - y \right| \ .$$

The first of these equations gives $b_n = -a_n$, while applying Fourier's formula

$$A_i = \int_a^b f \phi_i \, dx \bigg/ \int_a^b \phi_i^2 \, dx \ ,$$

where

$$\begin{aligned} \int_0^{\pi} \sin^2(ny) \, dy &= \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2} \cos(2ny) \right) \, dy \\ &= \frac{\pi}{2} - \frac{1}{4n} \sin(2ny) \bigg|_0^{\pi} = \frac{\pi}{2} \end{aligned}$$

to the second gives

$$\begin{aligned} 2a_n \sinh(n\pi) &= \frac{2}{\pi} \left(\int_0^{\pi/2} y \sin(ny) \, dy + \int_{\pi/2}^{\pi} (\pi - y) \sin(ny) \, dy \right) \\ &= \frac{2}{\pi} \left(-\frac{1}{n} \cos(ny) y \bigg|_0^{\pi/2} + \frac{1}{n} \int_0^{\pi/2} \cos(ny) \, dy \right. \\ &\qquad \left. - \frac{1}{n} \cos(ny)(\pi - y) \bigg|_{\pi/2}^{\pi} - \frac{1}{n} \int_{\pi/2}^{\pi} \cos(ny) \, dy \right) \\ &= \frac{2}{\pi} \left(-\frac{\pi}{2n} \cos(n\pi/2) + \frac{1}{n^2} (\sin(n\pi/2) - \sin(0)) \right. \\ &\qquad \left. + \frac{\pi}{2n} \cos(n\pi/2) - \frac{1}{n^2} (\sin(n\pi) - \sin(n\pi/2)) \right) \\ &= \frac{4}{\pi n^2} \sin(n\pi/2) \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{\pi n^2} & \text{if } n \ = \ 4m+1 \\ -\frac{4}{\pi n^2} & \text{if } n \ = \ 4m+3 \end{cases} \end{aligned}$$

Hence,

$$u_1(x,y) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}\frac{\sinh((2k+1)x)}{\sinh((2k+1)\pi)}\sin((2k+1)y)\ .$$

In identical fashion, we have

$$u_2(x,y) = \sum_{n=1}^{\infty} a_n \sinh(ny)\sin(nx)$$

where

$$\begin{aligned}
a_n \sinh(n\pi) &= \frac{2}{\pi}\int_0^{\pi} x^2(\pi - x)\sin(nx)\,dx \\
&= -\frac{2}{n\pi}\cos(nx)x^2(\pi - x)\Big|_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}(2\pi x - 3x^2)\cos(nx)\,dx \\
&= \frac{2}{n^2\pi}\sin(nx)(2\pi x - 3x^2)\Big|_0^{\pi} - \frac{2}{n^2\pi}\int_0^{\pi}(2\pi - 6x)\sin(nx)\,dx \\
&= \frac{2}{n^3\pi}\cos(nx)(2\pi - 6x)\Big|_0^{\pi} + \frac{12}{n^3\pi}\int_0^{\pi}\cos(nx)\,dx \\
&= -\frac{8}{n^3}\cos(n\pi) - \frac{4}{n^3} + \frac{12}{n^4\pi}(\sin(n\pi) - 0) \\
&= -\frac{4}{n^3}(1 + 2(-1)^n)
\end{aligned}$$

Hence,

$$u_2(x,y) = -4\sum_{n=1}^{\infty}\frac{1+2(-1)^n}{n^3}\frac{\sinh(ny)}{\sinh(n\pi)}\sin(nx)\ .$$

The solution of the original problem is $u_1(x,y) + u_2(x,y)$.