MATH 3403
Week 4

Semi-infinite string with moving endpoint.

Suppose now that instead of being located at $x = 0$, the endpoint is moving with velocity U along the line $x = Ut$.

There are three cases to consider;(i) $U > c$, (ii) $-c < U < c$, (iii) $U < -c$.

Firstly, $U > c$. In this case the endpoint is moving faster than the speed c at which signals are transmitted along the string. Therefore the behaviour of the endpoint has no effect on the solution, and we have

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$$

throughout the region $0 < Ut < x < \infty$.

In the second case, when $-c < U < c$, D'Alembert's form provides a solution in the region $0 < ct < x < \infty$. In the region $Ut < x < ct$, the function $\psi(x + ct)$ is defined by the initial conditions on $0 < x < \infty$, and we need only supply one further piece of *boundary* data on the line $x = Ut$ in order to specify the solution. In this respect the solution resembles the fixed end-point cases which we have studied above.

Finally, if $U < -c$, the endpoint is receding faster than the speed at which signals can be transmitted. The initial data can only specify $\psi(x+ct)$ in the region $-ct \leq x$, so that in order to determine the solution in the region $Ut < x < -ct$, we need to specify two pieces of data (equivalent to the values of ϕ and ψ) along the line $x = Ut$.

We say that the boundary $x = Ut$ is **spacelike** in this case. A *spacelike* boundary is characterised by the condition that the characteristics $x + ct = x_0 + ct_0$ and $x - ct = x_0 - ct_0$ through a point (x_0, t_0) on the boundary both lie on the same side of the boundary for $t > t_0$.

In contrast, the boundary in the second case is called **timelike**. The characteristics through a point (x_0, t_0) on a *timelike* boundary emerge on opposite sides of the boundary for $t > t_0$.

The boundary in case one is also **spacelike**. However, in this case it intersects both families of characteristics emanating from the initial line $t = 0$, and any attempt to specify data along the line would lead to a contradiction.

Example.

If we choose $f(x) = \sin x$ and $g(x) = 0$, then D'Alembert's solution is

$$u(x,t) = \frac{1}{2}(\sin(x-ct) + \sin(x+ct)) = \sin x \cos ct\ .$$

If $U > c$, then this solution exists on the line $x = Ut$, and defines the value of u completely.

If $-c < U < c$, then in the region $Ut < x < ct$, $x - ct < 0$ but $x + ct > 0$. Therefore $f(x - ct)$ is not defined while $f(x + ct)$ is.

Suppose that we add the condition $u(Ut, t) = 0$, in order to define the solution in this region.

$$\begin{aligned}
u(x,t) &= \phi(x-ct)+\psi(x+ct)\\
&= \phi(x-ct)+\frac{1}{2}\sin(x+ct)\\
u(Ut,t) &= \phi((U-c)t)+\frac{1}{2}\sin((U+c)t)=0\\
\phi((U-c)t) &= -\frac{1}{2}\sin((U+c)t)\\
\phi(y) &= -\frac{1}{2}\sin\left(\frac{U+c}{U-c}y\right)\\
\phi(x-ct) &= -\frac{1}{2}\sin\left(\frac{c+U}{c-U}(ct-x)\right)\\
u(x,t) &= \frac{1}{2}\sin(x+ct)-\frac{1}{2}\sin\left(\frac{c+U}{c-U}(ct-x)\right)
\end{aligned}$$

Finally, consider $U < -c$. In this case both $x-ct$ and $x+ct$ are negative in the region $Ut < x < -ct$, so that no information from the initial conditions on $0 < x < \infty$ can influence the solution here. Therefore we need to provide two pieces of data in order to specify the solution in this region. For simplicity, let us specify $u(Ut,t) = u_t(Ut,t) = 0$, so that the functions $\phi(x-ct)$ and $\psi(x+ct)$ are both identically zero in this wedge.

The solution in the wedge $-ct < x < ct$ is obtained by extrapolating the solution $\phi(x-ct) \equiv 0$ from the left and the solution $\psi(x+ct) \equiv \frac{1}{2}\sin(x+ct)$ from the right. Hence the full solution in this case is

$$\begin{aligned}
u(x,t) &= 0 && Ut < x < -ct\\
&= \frac{1}{2}\sin(x+ct) && -ct < x < ct\\
&= \sin x\cos ct && x > ct
\end{aligned}$$

The inhomogeneous wave equation.

$$u_{xx}-c^{-2}u_{tt}=f(x,t)$$

In common with other linear inhomogeneous equations, the general solution consists of the sum of a **particular solution** and the **complementary function**.

If $U(x,t)$ is any solution of $u_{xx}-c^{-2}u_{tt}=f(x,t)$, then writing $u(x,t)=U(x,t)+v(x,t)$, we have

$$\begin{gathered}
u_{xx}-c^{-2}u_{tt}=f(x,t)\\
U_{xx}+v_{xx}-c^{-2}(U_{tt}+v_{tt})=f(x,t)\\
v_{xx}-c^{-2}v_{tt}=f(x,t)-(U_{xx}-c^{-2}U_{tt})=0\ .
\end{gathered}$$

The problem of solving the inhomogeneous wave equation therefore breaks down into the two simpler problems of finding **any** solution of the inhomogeneous equation and then using the techniques we have already developed to solve the remaining homogeneous initial/boundary value problem.

For the inhomogeneous wave equation there is an algorithm for calculating a particular solution, which we shall derive shortly. However, we first deal with two related cases in which a particular solution can be obtained very simply. These cases occur when the function $f(x,t)$ is either a function of x alone or a function of t alone.

In the first case we can find a particular solution U as a function of x alone by solving

$$U'' = f(x) \ .$$

For example, a particular solution of $u_{xx} - c^{-2}u_{tt} = x$ is given by solving

$$U'' = x \quad ; \quad U = \frac{1}{6}x^3 \ .$$

(Since any solution will do, we do not need to worry about the arbitrary constants of integration.)

In similar fashion, when f is a function of t alone, we find a particular solution $U(t)$ by solving

$$-c^{-2}\ddot{U} = f(t) \ ; \ \ddot{U} = -c^2 f(t) \ .$$

Hence, a particular solution of $u_{xx} - c^{-2}u_{tt} = \sin t$ is $U = c^2 \sin t$.

This approach can be extended to the case $f(x,t) = f_1(x) + f_2(t)$ by using the principle of superposition.

If $U_1''(x) = f_1(x)$ and $\ddot{U}_2(t) = -c^2 f_2(t)$, then $U(x,t) = U_1(x) + U_2(t)$ is a particular solution of the inhomogeneous equation

$$u_{xx} - c^{-2}u_{tt} = f_1(x) + f_2(t) \ .$$

Suppose now that we wish to solve the initial value problem

$$\begin{gathered} u_{xx} - c^{-2}u_{tt} = x \ ; \qquad -\infty < x < \infty \ , \ 0 < t < \infty \\ u(x,0) = 0 \ ; \ u_t(x,0) = 0 \ . \end{gathered}$$

We know that $\frac{1}{6}x^3$ is a particular solution of the inhomogeneous equation. It does not, however, satisfy the initial conditions. Therefore we set $u(x,t) = v(x,t) + \frac{1}{6}x^3$. The function v is a solution of the homogeneous wave equation which satisfies the initial conditions

$$\begin{gathered} u(x,0) = v(x,0) + \frac{1}{6}x^3 = 0 \ ; \ v(x,0) = -\frac{1}{6}x^3 \\ u_t(x,0) = v_t(x,0) + 0 = 0 \ ; v_t(x,0) = 0 \end{gathered}$$

Using D'Alembert's solution we obtain

$$v(x,t) = \frac{1}{2}\left(-\frac{1}{6}(x-ct)^3 - \frac{1}{6}(x+ct)^3\right)$$

$$= -\frac{1}{12}\left(x^3 - 3x^2ct + 3xc^2t^2 - c^3t^3 + x^3 + 3x^2ct + 3xc^2t^2 + c^3t^3\right)$$

$$v(x,t) = -\frac{1}{6}x^3 - \frac{1}{2}c^2xt^2$$

$$u(x,t) = -\frac{1}{2}c^2xt^2$$

Similarly, let us consider the initial/boundary value problem

$$u_{xx} - c^{-2}u_{tt} = \sin t \ ; \qquad 0 < x < \infty \ , \ 0 < t < \infty$$
$$u(x,0) = u_t(x,0) = 0 \ , \ 0 < x < \infty \ ; \ u(0,t) = 0 \ , \ 0 < t < \infty$$

Since $c^2 \sin t$ is a particular solution of the inhomogeneous equation we set $u(x,t) = v(x,t) + c^2 \sin t$. The function v is now the solution of the homogeneous initial/boundary value problem which satisfies

$$u(x,0) = v(x,0) + c^2 \sin(0) = 0 \ ; \ v(x,0) = 0$$
$$u_t(x,0) = v_t(x,0) + c^2 \cos(0) = 0 \ ; \ v_t(x,0) = -c^2$$
$$u(0,t) = v(0,t) + c^2 \sin t = 0 \ ; \ v(0,t) = -c^2 \sin t$$

In the region $x > ct$, D'Alembert's solution gives

$$v(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} (-c^2)\, ds = -c^2 t$$

so that $u(x,t) = c^2(\sin t - t)$.

In the region $0 < x < ct$

$$\begin{aligned} v(x,t) &= -c^2 \sin\left(t - \frac{x}{c}\right) + \frac{1}{2c}\int_{ct-x}^{ct+x} (-c^2)\, ds \\ &= -c^2\left(\sin\left(t - \frac{x}{c}\right) + \frac{x}{c}\right) \\ u(x,t) &= c^2\left(\sin t - \sin\left(t - \frac{x}{c}\right) - \frac{x}{c}\right) \end{aligned}$$

A general algorithm for finding a particular solution.

Given the inhomogeneous equation

$$u_{xx} - c^{-2}u_{tt} = f(x,t)$$

we will derive the particular solution which satisfies the homogeneous initial conditions $u(x,0) = u_t(x,0) = 0$. Note that $u_x(x,0) = 0$ also.

Introducing the characteristic variables $\xi = x - ct$ and $\eta = x + ct$, and setting $u(x,t) = U(\xi,\eta)$, the equation becomes

$$4U_{\xi\eta} = f\left(\frac{\xi+\eta}{2} \ , \ \frac{\eta-\xi}{2c}\right) = F(\xi,\eta)$$

from which we can derive the required solution by integration.

The initial line corresponds to the line $\xi = \eta$ in the transformed plane. Along this line we have $U = 0$ and $U_\eta = 0$. Combining these we obtain

$$\begin{aligned} U_{\xi\eta} &= \frac{1}{4}F(\xi,\eta) \\ U_\eta(\eta,\eta) - U_\eta(\xi_0,\eta) &= \int_{\xi_0}^{\eta} U_{\xi\eta}(\xi,\eta)\, d\xi \\ U_\eta(\xi_0,\eta) &= -\frac{1}{4}\int_{\xi_0}^{\eta} F(\xi,\eta)\, d\xi \\ U(\xi_0,\eta_0) - U(\xi_0,\xi_0) &= \int_{\xi_0}^{\eta_0} U_\eta(\xi_0,\eta)\, d\eta \\ U(\xi_0,\eta_0) &= -\frac{1}{4}\int_{\xi_0}^{\eta_0}\left(\int_{\xi_0}^{\eta} F(\xi,\eta)\, d\xi\right) d\eta \end{aligned}$$

The double integral is taken over the triangle bounded by the lines $\xi = \xi_0$, $\eta = \eta_0$ and $\xi = \eta$ in the transformed plane. This corresponds to the triangle bounded by the characteristics $x - ct = x_0 - ct_0$, $x + ct = x_0 + ct_0$ and the initial line $t = 0$ in the x - t plane. The Jacobian of the transformation is

$$\frac{\partial(\xi,\eta)}{\partial(x,t)} = \begin{vmatrix} 1 & -c \\ 1 & c \end{vmatrix} = 2c$$

so that in terms of the original co-ordinates the particular solution is

$$u(x_0,t_0) = -\frac{c}{2}\int_0^{t_0}\left(\int_{x_0-ct_0+ct}^{x_0+ct_0-ct} f(x,t)\,dx\right)dt$$

or equivalently

$$u(x,t) = -\frac{c}{2}\int_0^{t}\left(\int_{x-ct+c\tau}^{x+ct-c\tau} f(\sigma,\tau)\,d\sigma\right)d\tau$$

Example.

Determine the solution of the initial value problem

$$u_{xx} - c^{-2}u_{tt} = e^{-t}\sin x\ ,\ \ u(x,0) = u_t(x,0) = 0\ .$$

Substituting into the integral form we have:

$$\begin{aligned}\int_{x-ct+c\tau}^{x+ct-c\tau} e^{-\tau}\sin\sigma\,d\sigma &= -e^{-\tau}\cos\sigma\Big]_{x-ct+c\tau}^{x+ct-c\tau}\\ &= -e^{-\tau}\left(\cos(x+ct-c\tau) - \cos(x-ct+c\tau)\right)\\ &= 2e^{-\tau}\sin x\sin(c(t-\tau))\end{aligned}$$

$$\begin{aligned}\int_0^t 2e^{-\tau}\sin x\sin(c(t-\tau))\,d\tau &= 2e^{-t}\sin x\int_0^t e^{(t-\tau)}\sin(c(t-\tau))\,d\tau\\ = 2e^{-t}\sin x\int_0^t e^{s}\sin(cs)\,ds &= 2e^{-t}\sin x\left[\frac{1}{1+c^2}e^{s}(\sin cs - c\cos cs)\right]_0^t\\ &= \frac{2}{1+c^2}e^{-t}\sin x\left(e^{t}(\sin ct - c\cos ct) + c\right)\\ u(x,t) &= -\frac{c}{1+c^2}\sin x\left(\sin ct - c\cos ct + ce^{-t}\right)\end{aligned}$$

Checking, we have

$$\begin{aligned}u_{xx}(x,t) &= \frac{c}{1+c^2}\sin x\left(\sin ct - c\cos ct + ce^{-t}\right)\\ u_{tt}(x,t) &= \frac{c}{1+c^2}\sin x\left(c^2\sin ct - c^3\cos ct - ce^{-t}\right)\\ u_{xx} - c^{-2}u_{tt} &= \frac{1}{1+c^2}\sin x\left(c^2e^{-t} + e^{-t}\right) = \sin x e^{-t}\end{aligned}$$

Summary.

While this form for the particular solution has been derived for the case of an infinite string, it applies equally well in other cases provided the function $f(x,t)$ is extended onto the domain $-\infty < x < \infty$ outside the region of interest in some appropriate fashion, either by retaining the same functional form or defining $f \equiv 0$.

This form for the particular solution satisfies homogeneous initial conditions, so that the complementary function is essentially D'Alembert's solution. However, in semi-infinite and finite applications, the boundary conditions usually have to be modified to accommodate non-zero contributions from the particular solution.

The telegraph equation.

The wave equation which we have so far studied is unrealistic in so far as it ignores damping and dispersion of the wave.

A more realistic model is the *telegraph equation*

$$u_{xx} = \frac{1}{c^2} u_{tt} + a u_t + b u$$

where the term au_t represents the damping and the term bu the dispersion.

To solve this equation, we first eliminate the damping term.
If we write $u(x,t) = d(t)v(x,t)$, the equation becomes

$$\begin{aligned} d v_{xx} &= \frac{1}{c^2}\left(d''v + 2d'v_t + dv_{tt}\right) + a(d'v + dv_t) + bdv \\ v_{xx} &= \frac{1}{c^2} v_{tt} \\ &\quad + \left(\frac{1}{c^2}\frac{2d'}{d} + a\right) v_t \\ &\quad + \left(\frac{1}{c^2}\frac{d''}{d} + a\frac{d'}{d} + b\right) v \end{aligned}$$

and we can eliminate the coefficient of v_t by choosing

$$\begin{aligned} \frac{1}{c^2}\frac{2d'}{d} + a &= 0 \\ d' &= -\frac{ac^2}{2} d \\ d &= \exp\left(-\frac{ac^2}{2} t\right) \end{aligned}$$

Substituting this into the previous form gives

$$v_{xx} = \frac{1}{c^2} v_{tt} + \left(b - \frac{a^2c^2}{4}\right) v$$

which reduces to the wave equation when $b = a^2c^2/4$.
Therefore in this case the general solution of the telegraph equation is

$$u(x,t) = e^{-ac^2t/2}(\phi(x-ct) + \psi(x+ct))$$

which represents travelling waves whose shape (ϕ, ψ) is preserved but whos amplitude decays exponentially with time. We say that these waves are *relatively indistorted*.

Consequently, transmission lines are usually designed to ensure that this condition holds. The amplitude decay is accommodated by putting booster stations at regular intervals along the line.

Suppose that $k = b - a^2c^2/4 \neq 0$, and look for travelling wave solutions $\psi(x - \gamma t)$ of the equation

$$v_{xx} = \frac{1}{c^2} v_{tt} + kv \ .$$

Substituting into the equation we obtain

$$\psi'' = \frac{\gamma^2}{c^2}\psi'' + k\psi$$
$$\psi'' + \frac{kc^2}{\gamma^2 - c^2}\psi = 0$$

which has sinusoidal solutions when $k(\gamma^2 - c^2) > 0$, exponential solutions when $k(\gamma^2 - c^2) < 0$, but **no nontrivial solutions** when $\gamma^2 = c^2$.

This property of waves travelling at different speeds is called *dispersion*.

Depending on the sign of k, we have families of solutions

$$\sin(x - \gamma t), \cos(x - \gamma t)$$

for $0 \leq \gamma < c$ or $c < \gamma$, and we can derive more general solutions in the forms

$$v(x,t) = \int_0^c \left(a(\gamma)\cos(x - \gamma t) + b(\gamma)\sin(x - \gamma t)\right)\, d\gamma$$

$$v(x,t) = \int_c^\infty \left(a(\gamma)\cos(x - \gamma t) + b(\gamma)\sin(x - \gamma t)\right)\, d\gamma$$

Nonlinear wave equations.

We can also look for travelling wave solutions of the more general non-linear wave equation

$$u_{xx} - \frac{1}{c^2}u_{tt} = f(u, u_x, u_t)$$

If we set $u = \phi(x - \gamma t)$, we see that ϕ satisfies the equation

$$\left(1 - \frac{\gamma^2}{c^2}\right)\phi'' = f(\phi, \phi', -\gamma\phi')$$

which is a second order autonomous equation when $\gamma \neq c$.