MATH 3403

Week 3

The wave equation on an infinite domain.

In order to postpone the problems associated with end conditions, we will first consider the equation

$$u_{xx} = c^{-2} u_{tt}$$

in the region $-\infty < x < \infty$, $0 \le t < \infty$.

As we have already seen, the general solution of the wave equation is

$$u = \phi(x - ct) + \psi(x + ct) \ .$$

Let us consider the two parts of this general solution.

$$u = \phi(x - ct)$$

is a possible solution to the equation. At time $t = 0$ it represents a displacement $u = \phi(x)$. At a later fixed time the displacement is $u = \phi(x - ct_1)$, or $u = \phi(\xi)$, where $\xi = x - ct_1$ is a new co-ordinate obtained by translating the origin a distance ct_1 to the right. So the displacement retains the same shape as time changes, but moves to the right with velocity c.

Another way of putting this is as follows.

Take a particular value x_0. The displacement at x_0 at time $t = 0$ is the same as that at that point x at time t for which

$$x - ct = x_0 \qquad \text{or} \qquad x = x_0 + ct \ .$$

Thus the point x whose displacement is the same as that at x_0 at time $t = 0$ travels to the right according to the equation $x = x_0 + ct$, that is, with velocity

$$\frac{dx}{dt} = \frac{d}{dt}(x_0 + ct) = c \ .$$

We call $\phi(x - ct)$ a *travelling* or *progressive wave*, travelling to the right with velocity c. It is constant along the **characteristic** lines $x - ct$ constant.

Similarly, $\psi(x + ct)$ is a travelling wave travelling to the left with velocity c. In this case the **characteristics** are the lines $x + ct$ constant.

Therefore each solution

$$u(x,t) = \phi(x - ct) + \psi(x + ct)$$

is a superposition of two travelling waves, one travelling to the right and the other to the left, with the same velocity.

This solution involves two arbitrary functions. In order to determine $u(x,t)$ at some point (x_0, t_0) we need to specify the value of ϕ along the characteristic $x - ct = x_0 - ct_0$ and the value of ψ along the characteristic $x + ct = x_0 + ct_0$. Specificly we need to know ϕ when $x = x_0 - ct_0$, $t = 0$, and ψ when $x = x_0 + ct_0$, $t = 0$; i.e. two pieces of data at each point on the line $t = 0$.

We could merely specify the values of ϕ and ψ when $t = 0$. However, it is usual to specify the **Cauchy Data**; that is, the value of the initial displacement $u(x, 0)$ and the value of the initial velocity $u_t(x, 0)$ at each point of the initial line; since this accords with the usual physical reality.

D'Alembert's solution.

If we specify $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ for $-\infty < x < \infty$, then substituting the general solution for $u(x,t)$ gives

$$\begin{aligned}
\phi(x) + \psi(x) &= f(x) \\
-c\phi'(x) + c\psi'(x) &= g(x) \\
\text{so that} \qquad -c\phi(x) + c\psi(x) &= \int_0^x g(s)\,ds \; + k \\
2c\phi(x) &= cf(x) - \int_0^x g(s)\,ds \; - k \\
\phi(x-ct) &= \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)\,ds - \frac{k}{2c} \\
2c\psi(x) &= cf(x) + \int_0^x g(s)\,ds \; + k \\
\psi(x+ct) &= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds + \frac{k}{2c} \\
\phi(x-ct) + \psi(x+ct) &= \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds
\end{aligned}$$

(Note that the constant k of integration does not appear in the final solution. Hence there is no loss of generality in assuming $k = 0$. Note also that the choice of 0 as the lower end point of integration does not affect the answer.)

This formula

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

is usually referred to as *D'Alembert's solution*.

Solution structure.

The solution $u(x_0, t_0)$ at the point (x_0, t_0), $(t_0 > 0)$, is fully determined by the values of f at $(x_0 - ct_0)$ and $(x_0 + ct_0)$ and the values of g in the interval $[(x_0 - ct_0), (x_0 + ct_0)]$. This interval is known as the **domain of dependence** of the solution at (x_0, t_0).

Changing the values of f and g outside this interval do not change the value of the solution at (x_0, t_0).

Physically, this means that information cannot be transmitted at a speed greater than c. (For example, according to Einstein's theory of relativity, information cannot be transmitted faster than the speed of light.) If we fix on a particular value x_0, then the initial values at $x_0 \pm k$ only affect the solution at (x_0, t_0) if $t_0 \geq k/c$. In particular, if the function f is discontinuous at α, then this discontinuity propagates along the **characteristic** lines $x - ct = \alpha$ and $x + ct = \alpha$.

Complementary to the domain of dependence is the **range of influence**. The value of the solution and its derivatives at the point (x_0, t_0) can only influence the subsequent solution in the wedge shaped region bounded by the characteristics $x - ct = x_0 - ct_0$ and $x + ct = x_0 + ct_0$ for $t > t_0$. This region is referred to as the *range of influence*. If we consider the points in the interval $[a, b]$ on the initial line

$t = 0$, their *range of influence* is the region bounded by $x + ct = a$ and $x - ct = b$ for $t > 0$.

Finally, the triangular region bounded by the interval $[a, b]$ on the initial line and the two characteristics $x - ct = a$ and $x + ct = b$ is referred to as the **domain of determinancy** of the interval, since the value of the solution at any point in this triangle is completely determined if the initial data is known on the interval.

If we consider the special case $g \equiv 0$, the solution reduces to

$$u(x,t) = \frac{1}{2}(f(x - ct) + f(x + ct)) \ .$$

This form indicates that the solution consists of two waves of identical shape ($\frac{1}{2}f$ i.e. half the initial displacement) one of which $\frac{1}{2}f(x - ct)$ moves to the right with speed c and the other $\frac{1}{2}f(x + ct)$ moves to the left with speed c. When g does not vanish identically, the solution can still be represented as the two waves, $\phi(x - ct)$ moving to the right and $\psi(x + ct)$ moving to the left, but the waves no longer have the same form.

Examples.

We will illustrate these properties by considering two simple examples.

Firstly, consider

$$f(x) = \begin{cases} 1 \ , & |x| \le a \\ 0 \ , & |x| > a \end{cases} \qquad g(x) = 0 \ .$$

This represents an initial pulse of unit height centered at the origin.

The solution $u(x,t) = \frac{1}{2}(f(x - ct) + f(x + ct)$ represents two pulses of height $\frac{1}{2}$, one moving to the left and the other to the right. For $0 < t < \frac{a}{c}$ these two pulses are merged near the origin, but for $t > \frac{a}{c}$ they are distinct.

Notice that the function f is discontinuous at $x = \pm a$, and that the solution is discontinuous along the characteristics $x - ct = \pm a$ and $x + ct = \pm a$ which pass through these points.

Secondly, consider

$$f(x) = 0 \ ; \quad g(x) = \begin{cases} 1 \ , & |x| \le 1 \\ 0 \ , & |x| > 1 \end{cases}$$

which represents an impulsively struck string.

The solution is

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$$

which breaks down into the cases

$$u(x,t) = 0 \text{ if } x + ct \le -1 \ \text{ or } \ x - ct \ge 1$$

$$u(x,t) = \frac{1}{c} \text{ if } x - ct \le -1 \ \text{ and } \ x + ct \ge 1$$

$$u(x,t) = \frac{x + ct + 1}{2c} \text{ if } x - ct \le -1 \ \text{ and } \ -1 \le x + ct \le 1$$

$$u(x,t) = \frac{1 - x + ct}{2c} \text{ if } \ -1 \le x - ct \le 1 \ \text{ and } x + ct \ge 1$$

$$u(x,t) = t \text{ if } \ -1 \le x - ct < x + ct \le 1$$

Note that the solution is continuous although the initial data is discontinuous. (Integration smooths things out.) However, the slope of the solution is discontinuous along the characteristics $x \pm ct = \pm 1$.

The wave equation on a semi-infinite domain.

Suppose that we wish to solve

$$u_{xx} = c^{-2} u_{tt} \ , \ \ x \geq 0 \ , \ \ t \geq 0$$
$$u(x, 0) = f(x) \ , \ \ u_t(x, 0) = g(x) \ , \ \ x \geq 0$$

D'Alembert's formula

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

only provides a solution provided $x - ct \geq 0$, since f and g are only defined for non-negative values of their arguments.

This also accords with the previous discussion. The **domain of determinancy** of the interval $[0, \infty)$ is the region bounded by the positive x-axis and the line $x = ct$, $t \geq 0$, and in this region the solution is completely determined by f and g.

This leaves the problem of determining the solution in the region $0 \leq x < ct$, $t > 0$.

The general solution is (still) $u(x,t) = \phi(x - ct) + \psi(x + ct)$. The function ψ is constant along the characteristics $x + ct$ constant, so that for (x, t) in the region of interest we can extrapolate the solution from the domain of determinancy, and take

$$\psi(x+ct) = \frac{1}{2} f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds \ .$$

In particular, on the boundary $x = 0$, $t > 0$,

$$\psi(ct) = \frac{1}{2} f(ct) + \frac{1}{2c}\int_0^{ct} g(s)\,ds \ .$$

However, to determine ϕ when its argument is negative we require one further piece of data provided on some curve transverse to the characteristic field $x = ct + \gamma$ ($\gamma < 0$). Normally this takes the form of **boundary data** provided along the line $x = 0$.

For example, if we know $u(0, t) = h(t)$ for $t > 0$, then

$$u(0,t) = \phi(-ct) + \psi(ct) = h(t)$$
$$\phi(-ct) = h(t) - \frac{1}{2} f(ct) - \frac{1}{2c}\int_0^{ct} g(s)\,ds$$
$$\phi(y) = h\left(-\frac{y}{c}\right) - \frac{1}{2} f(-y) - \frac{1}{2c}\int_0^{-y} g(s)\,ds$$
$$\text{where } \ y < 0 \ \text{ is a dummy argument}$$
$$\phi(x-ct) = h\left(t - \frac{x}{c}\right) - \frac{1}{2} f(ct-x) - \frac{1}{2c}\int_0^{ct-x} g(s)\,ds$$
$$\text{when } \ x - ct < 0$$
$$u(x,t) = \phi(x-ct) + \psi(x+ct)$$
$$= h\left(t - \frac{x}{c}\right) + \frac{1}{2}(-f(ct-x) + f(x+ct)) + \frac{1}{2c}\int_{ct-x}^{ct+x} g(s)\,ds$$

In particular, when $h = 0$, the endpoint $x = 0$ is **fixed**. In this case $\phi(y) = -\psi(-y)$, so that the travelling wave $\phi(x - ct)$ propagating to the right from the point $x = 0$ for $t > 0$ represents the **inverted** reflection of the incident wave $\psi(x + ct)$. That is, at a fixed boundary point, incident waves are reflected inverted.

Looking at it another way, D'Alambert's solution for the infinite case

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

gives

$$u(0,t) = \frac{1}{2}(f(-ct) + f(ct)) + \frac{1}{2c}\int_{-ct}^{ct} g(s)\,ds\ .$$

This will give $u(0,t) = 0$, provided we define $f(-x) = -f(x)$, and $g(-x) = -g(x)$. This amounts to replacing the boundary condition $u = 0$ by the odd extension of the initial conditions. This procedure produces two wave forms in the left half of the line. One of these travels further to the left and does not affect the solution for $x > 0$. The other is, by virtue of the choice of $f(-x)$ and $g(-x)$, the inverse of the left travelling wave in the right half of the line. It moves to the right and produces the effect described above in the region $x > 0$.

Superposition of solutions.

The solution when $h(t) \neq 0$ is obtained by adding

$$u^*(x,t) = \begin{cases} h\left(t - \frac{x}{c}\right)\ , & 0 \le x < ct \\ 0\ , & x \ge ct \ge 0 \end{cases}$$

to the solution when $h(t) = 0$. We have this additive property because the wave equation is **linear** and **homogeneous**.

Indeed, if we define

$$u_1(x,t) = \frac{1}{2}\left(f(x+ct) + \operatorname{sgn}(x-ct) f(|x-ct|)\right)$$

and

$$u_2(x,t) = \frac{1}{2c}\int_{|x-ct|}^{x+ct} g(s)\,ds$$

then $u_1(x,t)$ is the solution of

$$u_{xx} = c^{-2}u_{tt}\ ,\ 0 \le x < \infty\ ,\ 0 \le t < \infty$$
$$u(x,0) = f(x)\ ,\ u_t(x,0) = 0\ ,\ u(0,t) = 0$$

$u_2(x,t)$ is the solution of

$$u_{xx} = c^{-2}u_{tt}\ ,\ 0 \le x < \infty\ ,\ 0 \le t < \infty$$
$$u(x,0) = 0\ ,\ u_t(x,0) = g(x)\ ,\ u(0,t) = 0$$

and $u^*(x,t)$ is the solution of

$$u_{xx} = c^{-2}u_{tt}\ ,\ 0 \le x < \infty\ ,\ 0 \le t < \infty$$
$$u(x,0) = 0\ ,\ u_t(x,0) = 0\ ,\ u(0,t) = h(t)\ .$$

Because of the linearity

$$\frac{\partial^2}{\partial x^2}(u_1 + u_2 + u^*) - c^{-2}\frac{\partial^2}{\partial t^2}(u_1 + u_2 + u^*)$$
$$= \left(\frac{\partial^2 u_1}{\partial x^2} - c^{-2}\frac{\partial^2 u_1}{\partial t^2}\right) + \left(\frac{\partial^2 u_2}{\partial x^2} - c^{-2}\frac{\partial^2 u_2}{\partial t^2}\right) + \left(\frac{\partial^2 u^*}{\partial x^2} - c^{-2}\frac{\partial^2 u^*}{\partial t^2}\right)$$

and because each function is a solution of the homogeneous equation $u_{xx} - c^{-2}u_{tt} = 0$, this expression reduces to 0 also, and $u_1 + u_2 + u^*$ is the solution of the wave equation in the first quadrant which satisfies the sum of the initial and boundary conditions of its component parts; namely

$$u(x, 0) = f(x)\ ,\ u_t(x, 0) = g(x)\ ,\ u(0, t) = h(t)\ .$$

This ability to construct solutions to complicated problems as the sums of solutions of simpler problems is an important feature of linear equations, to which we will return later.

Free boundary conditions.

The second common boundary condition is the **free** condition $u_x(0,t) = 0$.

$$\begin{aligned}
\text{If} \qquad & u(x,t) = \phi(x-ct) + \psi(x+ct)\ , \\
\text{then} \qquad & u_x(x,t) = \phi'(x-ct) + \psi'(x+ct) \\
& u_x(0,t) = \phi'(-ct) + \psi'(ct) = 0 \\
& \phi'(y) = -\psi'(-y) \\
& \int_0^y \phi'(s)\,ds = -\int_0^y \psi'(-s)\,ds = \int_0^{-y} \psi'(\sigma)\,d\sigma \\
& \phi(x-ct) - \phi(0) = \psi(ct-x) - \psi(0)
\end{aligned}$$

In order to determine the unknown constant $\phi(0)$, we assume (quite reasonably) that the solution is continuous across the line $x = ct$.

From D'Alembert's form,

$$u(ct,t) = \frac{1}{2}(f(0) + f(2ct)) + \frac{1}{2c}\int_0^{2ct} g(s)\,ds\ .$$

Using the expression for ψ,

$$\begin{aligned}
\psi(x+ct) &= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds \\
\phi(x-ct) &= \phi(0) + \frac{1}{2}f(ct-x) + \frac{1}{2c}\int_0^{ct-x} g(s)\,ds - \frac{1}{2}f(0) \\
u(x,t) &= \phi(x-ct) + \psi(x+ct)\ ;\ 0 \le x \le ct \\
u(ct,t) &= \phi(0) + \frac{1}{2}f(0) - \frac{1}{2}f(0) + \frac{1}{2}f(2ct) + \frac{1}{2c}\int_0^{2ct} g(s)\,ds \\
\text{therefore} \qquad & \phi(0) = \frac{1}{2}f(0) = \psi(0) \\
\text{and} \qquad & \phi(x-ct) = \psi(ct-x) \\
& \text{so that, when} \quad 0 \le x < ct\ ,
\end{aligned}$$

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(ct-x)) + \frac{1}{2c}\left(\int_0^{ct-x} g(s)\,ds + \int_0^{x+ct} g(s)\,ds\right)\ .$$

In this case the reflected wave ϕ has the same form as the incident wave ψ; it is not inverted at the end point.

This result can also be achieved by taking the even extension of the initial conditions; $f(-x) = f(x)$, $g(-x) = g(x)$; in place of the boundary condition at $x = 0$.

The general free endpoint condition.

Suppose that we have to solve the problem

$$\begin{gathered} u_{xx} = c^{-2}u_{tt}\ , \qquad 0 < x < \infty\ ,\ 0 < t < \infty\ ; \\ u(x,0) = f(x)\ ,\ u_t(x,0) = g(x)\ , \quad 0 < x < \infty\ ; \\ u_x(0,t) = h(t)\ , \quad 0 < t < \infty\ . \end{gathered}$$

In the region $x > ct > 0$, D'Alembert's form of the solution

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

still holds.

In the region $0 < x < ct$, we can express the solution as the sum of the solutions of the wave equation

(a) with initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$,
and boundary condition $u_x(0,t) = 0$;

(b) with initial conditions $u(x,0) = u_t(x,0) = 0$,
and boundary condition $u_x(0,t) = h(t)$.

The solution (a) has been found earlier as

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(ct-x)) + \frac{1}{2c}\left(\int_0^{ct-x} g(s)\,ds + \int_0^{x+ct} g(s)\,ds\right)\ .$$

For the solution (b) we proceed as before, looking for a solution in the form

$$u(x,t) = \phi(x-ct) + \psi(x+ct).$$

However, because of the homogeneous initial conditions, $\psi \equiv 0$, and we are left with

$$\begin{aligned} u(x,t) &= \phi(x-ct) \\ u_x(x,t) &= \phi'(x-ct) \\ u_x(0,t) &= \phi'(-ct) = h(t) \\ \phi'(y) &= h\left(-\frac{y}{c}\right) \\ \phi(\eta) &= \int_0^{\eta} h\left(-\frac{y}{c}\right)\,dy \end{aligned}$$

(where the continuity condition $u(ct,t) = 0$ has been used)

$$\phi(x-ct) = \int_0^{x-ct} h\left(-\frac{y}{c}\right)\,dy = -c\int_0^{t-x/c} h(s)\,ds$$

The full solution in the region $0 < x < ct$ is then the sum of these two partial solutions.

Transmission problems.

Problems of this type arise when a wave impinges on some discontinuity in a medium.

Typically in the one dimensional case we might consider a wave passing along a cable which meets a point at which the structure of the cable changes.

Another example is light impinging on a pane of glass.

Because the wave speed is different in the two media, the problem is in essence two semi-infinite wave problems, with the solutions coupled across the interface.

The equations to be solved are

$$u_{xx} = \frac{1}{c_1^2} u_{tt} \text{ for } x < 0$$
$$u_{xx} = \frac{1}{c_2^2} u_{tt} \text{ for } x > 0$$

where $c_1 \neq c_2$.

The general solution of the problem therefore has the form

$$u(x,t) = \phi_1(t - x/c_1) + \psi_1(t + x/c_1) \text{ for } x < 0$$
$$u(x,t) = \phi_2(t - x/c_2) + \psi_2(t + x/c_2) \text{ for } x > 0$$

and initial data $u(x,0)$, $u_t(x,0)$, will specify the first solution uniquely for $x \leq -c_1 t$ and the second solution for $x \geq c_2 t$.

However, in order to determine $\psi_1(t + x/c_1)$ for $-c_1 t < x < 0$, and $\phi_2(t - x/c_2)$ for $0 < x < c_2 t$, we need an interface condition between the two solutions along the line $x = 0$.

The most common such condition is to require that the solution and its space derivative be continuous along this line. That is

$$u(0-,t) = u(0+,t)$$
$$u_x(0-,t) = u_x(0+,t) \ .$$

We model the incoming wave by a function

$$F(t - x/c_1)$$

which represents a right moving wave in the first medium.

We assume that it has a sharp front, so that at time $t = 0$ the region $x > 0$ is undisturbed.

This corresponds to the initial conditions

$$\left\{ \begin{array}{l} u(x,0) = F(-x/c_1) \\ u_t(x,0) = F'(-x/c_1) \end{array} \right. \quad x < 0$$
$$u(x,0) = u_t(x,0) = 0 \ x > 0$$

The solution for $x < -c_1 t$ is $u = F(t - x/c_1)$ and for $x > c_2 t$, $u = 0$.

In the region $-c_1 t < x < 0$,

$$u(x,t) = F(t - x/c_1) + \psi_1(t + x/c_1)$$

for some function ψ_1,

and in the region $0 < x < c_2 t$,

$$u(x,t) = \phi_2(t - x/c_2) \ .$$

The function ϕ_2 is called the transmitted wave, and the function ψ_1 the reflected wave.

Our interface conditions now give

$$F(t) + \psi_1(t) = \phi_2(t)$$

$$\text{and} \quad -\frac{1}{c_1}F'(t) + \frac{1}{c_1}\psi_1'(t) = -\frac{1}{c_2}\phi_2'(t)$$

$$\text{so that } F'(t) - \psi_1'(t) = \frac{c_1}{c_2}\phi_2'(t)$$

$$F(t) - \psi_1(t) = \frac{c_1}{c_2}\phi_2(t)$$

since all the functions vanish at the origin.

Hence

$$2F = \left(1 + \frac{c_1}{c_2}\right)\phi_2$$

$$\phi_2 = \frac{2c_2}{c_1 + c_2}F$$

$$\text{and} \quad \left(1 - \frac{c_1}{c_2}\right)F = \left(\frac{c_1}{c_2} + 1\right)\psi_1$$

$$\psi_1 = \frac{c_2 - c_1}{c_1 + c_2}F$$

The coefficient

$$T = \frac{2c_2}{c_1 + c_2}$$

is called the *transmission coefficient*,

and the the coefficient

$$R = \frac{c_2 - c_1}{c_1 + c_2}$$

is called the *reflection coefficient*.

Note that the transmitted wave retains the same shape as the incident wave, althought the amplitude and velocity of propogation alter.

Also, if $c_2 = c_1$, which corresponds to no interface, the incident wave continues unaltered and there is no reflected wave.