MATH 3403

WEEK 2

Classification of Second Order Operators.

The technique of simplifying the form of a linear differential operator

$$\{\text{setting} \quad a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} \equiv \frac{du}{dt} \}$$

which is the basis of the solution process for first order linear partial differential equations, is also applied to second order linear operators in two independent variables.

Consider the operator

$$a(x,y)u_{xx} + 2h(x,y)u_{xy} + b(x,y)u_{yy} \ .$$

If we introduce new variables $\xi(x,y)$ and $\eta(x,y)$, then

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x \text{ The Chain Rule} \\
u_y &= u_\xi \xi_y + u_\eta \eta_y \\
u_{xx} &= \frac{\partial}{\partial x}(u_\xi \xi_x + u_\eta \eta_x) \\
&= \frac{\partial u_\xi}{\partial x}\xi_x + u_\xi \xi_{xx} + \frac{\partial u_\eta}{\partial x}\eta_x + u_\eta \eta_{xx} \\
&= (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x)\xi_x + u_\xi \xi_{xx} \\
&\quad + (u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x)\eta_x + u_\eta \eta_{xx} \\
&= u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}(\eta_x)^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy} \\
u_{yy} &= u_{\xi\xi}(\xi_y)^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}(\eta_y)^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}
\end{aligned}$$

Substituting these expansions into the operator, we obtain

$$\begin{aligned}
&u_{\xi\xi}\left(a(x,y)(\xi_x)^2 + 2h(x,y)\xi_x\xi_y + b(x,y)(\xi_y)^2\right) \\
&+2u_{\xi\eta}\left(a(x,y)\xi_x\eta_x + h(x,y)(\xi_x\eta_y + \xi_y\eta_x) + b(x,y)\xi_y\eta_y\right) \\
&+u_{\eta\eta}\left(a(x,y)(\eta_x)^2 + 2h(x,y)\eta_x\eta_y + b(x,y)(\eta_y)^2\right) \\
&+u_\xi\left(a(x,y)\xi_{xx} + 2h(x,y)\xi_{xy} + b(x,y)\xi_{yy}\right) \\
&+u_\eta\left(a(x,y)\eta_{xx} + 2h(x,y)\eta_{xy} + b(x,y)\eta_{yy}\right)
\end{aligned}$$

We derive the **canonical form** (*canonical* is the fancy mathematical word for *standard*) by choosing the characteristic variables ξ and η so that they satisfy

$$\begin{aligned}
a(x,y)(\xi_x)^2 + 2h(x,y)\xi_x\xi_y + b(x,y)(\xi_y)^2 &= 0 \\
a(x,y)(\eta_x)^2 + 2h(x,y)\eta_x\eta_y + b(x,y)(\eta_y)^2 &= 0
\end{aligned}$$

independently (if possible).

There are three possible outcomes:
Firstly, the quadratic equation

$$aX^2 + 2hX + b = 0$$

may have two independent real solutions $X_1 = \alpha(x, y)$ and $X_2 = \beta(x, y)$. (This result may depend on the values of x and y.) In this case the operator is said to be **hyperbolic**.

It is possible to define new independent variables ξ and η as solutions of the first order linear partial differential equations

$$\xi_x = \alpha(x, y)\xi_y$$
$$\eta_x = \beta(x, y)\eta_y$$

For example, consider the operator

$$u_{xx} + 3u_{xy} + 2u_{yy} \ .$$

In this case, the quadratic equation is $X^2 + 3X + 2 = 0$, which has the solutions $X = -1$ and $X = -2$ for all values of x and y. This operator is therefore *hyperbolic* throughout the plane.

We can therefore introduce new independent variables which satisfy

(a) $$\xi_x + \xi_y = 0$$

which we can solve by using the characteristic equations with arbitrary initial conditions.

$$\frac{dx}{dt} = 1 \ , \ x = t + a(s)$$
$$\frac{dy}{dt} = 1 \ , \ y = t + b(s) \ , \ y - x = c(s)$$
$$\frac{d\xi}{dt} = 0 \ , \ \xi = \phi(s) \ = \ f(y - x)$$

In this solution, a, b, c, ϕ and f stand for arbitrary functions whose precise form is irrelevant.

For simplicity, we usually choose $\xi = x - y$.

The second new variable satisfies

(b) $$\eta_x + 2\eta_y = 0$$
$$\frac{dx}{dt} = 1 \ , \ x = t + a(s)$$
$$\frac{dy}{dt} = 2 \ , \ y = 2t + b(s) \ = \ 2x + c(s)$$
$$\frac{d\eta}{dt} = 0 \ , \ \eta = \phi(s) \ = \ f(y - 2x)$$

Again, it is usual to choose a simple form like $\eta = y - 2x$.

For these variables, $\xi_x = 1$, $\xi_y = -1$, $\xi_{xx} = \xi_{xy} = \xi_{yy} = 0$, $\eta_x = -2$, $\eta_y = 1$, and $\eta_{xx} = \eta_{xy} = \eta_{yy} = 0$.

Substituting these values into the form of the expansion of the operator we obtain

$$\begin{gathered}
u_{\xi\xi}(1(1)^2 + 3(1)(-1) + 2(-1)^2) \\
+2u_{\xi\eta}(1(1)(-2) + 1.5((1)(1) + (-1)(-2)) + 2(-1)(1)) \\
+u_{\eta\eta}(1(-2)^2 + 3(-2)(1) + 2(1)^2) \\
+u_\xi(1(0) + 3(0) + 2(0)) \\
+u_\eta(1(0) + 3(0) + 2(0)
\end{gathered}$$

which reduces to the simple canonical form $u_{\xi\eta}$.

With this change of variables, the second order linear partial differential equation

$$u_{xx} + 3u_{xy} + 2u_{yy} = 1$$

reduces to the equation

$$u_{\xi\eta} = 1$$

whose general solution is

$$u = \xi\eta + f(\xi) + g(\eta) \ = \ -2x^2 + 3xy - y^2 + f(x-y) + g(y-2x) \ .$$

Another more complicated example. Consider the operator

$$3y^2 u_{xx} - 4xy u_{xy} + x^2 u_{yy} \ .$$

The quadratic equation $3y^2X^2 - 4xyX + x^2 = 0$ has the solutions $X = x/3y$ and $X = x/y$, which are distinct solutions provided $x \neq 0$ or $y \neq 0$. Therefore this equation is hyperbolic in each of the four quadrants, but not on the axes. {Note that when $x = 0$, the operator reduces to $3y^2u_{xx}$, while when $y = 0$ it reduces to x^2u_{yy} . These simple forms are classified as **parabolic**.}

Our new variables for the hyperbolic cases satisfy the equations

(a) $$\xi_x = \frac{x}{3y}\xi_y \ ; \ 3y\xi_x - x\xi_y = 0$$

(b) $$\eta_x = \frac{x}{y}\eta_y \ ; \ y\eta_x - x\eta_y = 0$$

From the first of these we obtain

$$\begin{gathered}
\frac{dx}{dt} = 3y \ ; \ \frac{dy}{dt} = -x \\
x\,dx \ + \ 3y\,dy \ = 0 \ ; \ x^2 + 3y^2 \ = \ c(s) \\
\frac{d\xi}{dt} = 0 \ ; \ \xi = \ \phi(s) \ = \ f(x^2 + 3y^2)
\end{gathered}$$

Let us take $\xi = x^2 + 3y^2$.

From the second we obtain, in similar fashion,

$$\frac{dx}{dt} = y \; ; \; \frac{dy}{dt} = -x$$
$$x\,dx \; + \; y\,dy \; = 0 \; ; \; x^2 + y^2 = c(s)$$
$$\frac{d\eta}{dt} = 0 \; ; \; \eta = \; \phi(s) \; = \; f(x^2 + y^2)$$

For the other variable we take $\eta = x^2 + y^2$.

For these variables, $\xi_x = 2x$, $\xi_y = 6y$, $\xi_{xx} = 2$, $\xi_{xy} = 0$ and $\xi_{yy} = 6$, while $\eta_x = 2x$, $\eta_y = 2y$, $\eta_{xx} = 2$, $\eta_{xy} = 0$ and $\eta_{yy} = 2$.

Substituting these values into the operator we obtain

$$\begin{gathered} u_{\xi\xi}(3y^2(2x)^2 - 4xy(2x)(6y) + x^2(6y)^2) \\ +2u_{\xi\eta}(3y^2(2x)(2x) - 2xy((2x)(2y) + (6y)(2x)) + x^2(6y)(2y)) \\ +u_{\eta\eta}(3y^2(2x)^2 - 4xy(2x)(2y) + x^2(2y)^2) \\ +u_{\xi}(3y^2(2) - 4xy(0) + x^2(6)) \\ +u_{\eta}(3y^2(2) - 4xy(0) + x^2(2) \end{gathered}$$

which reduces to

$$-16x^2y^2u_{\xi\eta} + (6x^2 + 6y^2)u_{\xi} + (2x^2 + 6y^2)u_{\eta}$$

From the expressions for ξ and η, we have $\xi - \eta = 2y^2$, $\xi - 3\eta = -2x^2$, so that in terms of the new variables the form is

$$4(\xi - \eta)(\xi - 3\eta)u_{\xi\eta} + 6\eta u_{\xi} + 2\xi u_{\eta}$$

Elliptic systems.

When the quadratic equation

$$aX^2 + 2hX + b = 0$$

has two real roots, the second order operator

$$a(x, y)u_{xx} + 2h(x, y)u_{xy} + b(x, y)u_{yy}$$

is hyperbolic.

The other common case is when the roots of this equation form a complex conjugate pair $X = \alpha(x, y) \pm i\beta(x, y)$. When this occurs, we say that the operator is **elliptic**.

We can follow the same approach as for the hyperbolic operator and introduce new variables ζ and ζ^* as the (conjugate) complex solutions of the equations

$$\zeta_x = (\alpha(x, y) + i\beta(x, y))\zeta_y$$
$$\zeta^*_x = (\alpha(x, y) - i\beta(x, y))\zeta^*_y$$

If we do so, the operator reduces (after simplification) to the form

$$u_{\zeta\zeta^*} + \quad \text{lower order terms}$$

We convert this to its canonical form by taking for ξ and η the real and imaginary parts of ζ, so that

$$\begin{aligned}
\xi &= \frac{1}{2}(\zeta + \zeta^*) \\
\eta &= \frac{1}{2i}(\zeta - \zeta^*) \\
\frac{\partial u}{\partial \zeta^*} &= \frac{1}{2}\frac{\partial u}{\partial \xi} - \frac{1}{2i}\frac{\partial u}{\partial \eta} \\
\frac{\partial^2 u}{\partial \zeta \partial \zeta^*} &= \frac{1}{2}\left(\frac{1}{2}\frac{\partial^2 u}{\partial \xi^2} + \frac{1}{2i}\frac{\partial^2 u}{\partial \eta \partial \xi}\right) \\
&\quad - \frac{1}{2i}\left(\frac{1}{2}\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{1}{2i}\frac{\partial^2 u}{\partial \eta^2}\right) \\
&= \frac{1}{4}\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right)
\end{aligned}$$

which is a multiple of the **Laplacean operator**.

For example, consider the operator

$$u_{xx} + 2u_{xy} + 2u_{yy} \ .$$

In this case the quadratic equation is $X^2 + 2X + 2 = 0$, which has the solutions $X = -1 \pm i$. This operator is therefore *elliptic* throughout the plane.

The complex variable ζ satisfies the first order equation

$$\zeta_x + (1 - i)\zeta_y = 0 \ .$$

The characteristic equations give

$$\begin{aligned}
\frac{dx}{dt} &= 1 \ , \ x = t + a(s) \\
\frac{dy}{dt} &= 1 - i \ , \ y = (1 - i)t + b(s) \ , \ y - (1 - i)x = c(s) \\
\frac{d\zeta}{dt} &= 0 \ , \ \zeta = \phi(s) = f(y - (1 - i)x)
\end{aligned}$$

Choosing the simplest form; $\zeta = (y - x) + ix$; gives the new variables $\xi = y - x$, $\eta = x$.

For these variables, $\xi_x = -1$, $\xi_y = 1$, $\eta_x = 1$ and $\eta_y = 0$. Substituting these into the expansion of the operator, we obtain

$$\begin{aligned}
&u_{\xi\xi}(1(-1)^2 + 2(-1)(1) + 2(1)^2) \\
&+2u_{\xi\eta}(1(-1)(1) + 1((-1)(0) + (1)(1)) + 2(1)(0)) \\
&+u_{\eta\eta}(1(1)^2 + 2(1)(0) + 2(0)^2) \\
&= u_{\xi\xi} + u_{\eta\eta}
\end{aligned}$$

Finally in this section, consider the second order linear partial differential equation

$$yu_{xx} + u_{yy} = 0 \ ,$$

which is known as the **Tricomi equation**.

If $y < 0$, the quadratic equation $yX^2 + 1 = 0$ has the two distinct real solutions $X = \pm 1/\sqrt{|y|}$, so that the equation is *hyperbolic*. The characteristic variables ξ, η satisfy respectively

$$\begin{aligned}
\xi_x - \frac{1}{\sqrt{|y|}}\xi_y &= 0 \\
\frac{dx}{dt} = 1 \ , \ \frac{dy}{dt} &= -\frac{1}{\sqrt{-y}} \\
\sqrt{-y}\,\frac{d(-y)}{dt} &= \frac{dx}{dt} \\
\frac{2}{3}(-y)^{3/2} &= x + c(s) \\
\xi = \phi(s) &= f\left(x - \frac{2}{3}(-y)^{3/2}\right)
\end{aligned}$$

and

$$\begin{aligned}
\eta_x + \frac{1}{\sqrt{|y|}}\eta_y &= 0 \\
\eta &= f\left(x + \frac{2}{3}(-y)^{3/2}\right)
\end{aligned}$$

Taking the obvious variables $\xi = x - \frac{2}{3}(-y)^{3/2}$ and $\eta = x + \frac{2}{3}(-y)^{3/2}$,
(so that $x = \frac{1}{2}(\xi + \eta)$ and $(-y)^{3/2} = \frac{3}{4}(\eta - \xi)$)
we have $\xi_x = 1$, $\xi_y = \sqrt{-y}$, $\xi_{xx} = \xi_{xy} = 0$, $\xi_{yy} = -1/(2\sqrt{-y})$,
and $\eta_x = 1$, $\eta_y = -\sqrt{-y}$, $\eta_{xx} = \eta_{xy} = 0$, $\eta_{yy} = 1/(2\sqrt{-y})$.
In terms of these variables the operator becomes

$$\begin{aligned}
u_{\xi\xi}(y + (-y)) + 2u_{\xi\eta}(y + y) + u_{\eta\eta}(y + (-y)) \\
+u_\xi\left(-\frac{1}{2\sqrt{-y}}\right) + u_\eta\left(\frac{1}{2\sqrt{-y}}\right)
\end{aligned}$$

so that the equation is

$$\begin{aligned}
4yu_{\xi\eta} &= \frac{1}{2\sqrt{-y}}(u_\xi - u_\eta) \\
u_{\xi\eta} &= \frac{1}{6}\frac{u_\xi - u_\eta}{\xi - \eta}
\end{aligned}$$

When $y > 0$, the roots of the quadratic equation $yX^2 + 1 = 0$ are the complex conjugate pair $X = \pm i/\sqrt{y}$, and the equation is *elliptic*. The complex canonical variable ζ satisfies the equation

$$\zeta_x - \frac{i}{\sqrt{y}}\zeta_y = 0$$
$$\frac{dx}{dt} = 1 \ , \ \frac{dy}{dt} = \frac{-i}{\sqrt{y}}$$
$$\sqrt{y}\, dy = -i\, dx$$
$$\frac{2}{3}y^{3/2} + ix = c(s) = \zeta$$

The corresponding real canonical variables are $\xi = \frac{2}{3}y^{3/2}$ and $\eta = x$.

The non-zero partial derivatives are $\xi_y = \sqrt{y}$, $\xi_{yy} = \frac{1}{2}y^{-1/2}$ and $\eta_x = 1$. Substituting these into the equation gives

$$y\, u_{\xi\xi} + y\, u_{\eta\eta} + \frac{1}{2\sqrt{y}}u_\xi = 0$$
$$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{3\xi}u_\xi = 0$$

Parabolic systems.

When the roots of the equation

$$aX^2 + 2hX + b = 0$$

are (real and) equal, we say that the operator

$$a(x, y)u_{xx} + 2h(x, y)u_{xy} + b(x, y)u_{yy}$$

is **parabolic**.

Most commonly this occurs along the boundaries of regions in which the operator is elliptic or hyperbolic. In the previous example, the Tricomi equation is *parabolic* when $y = 0$. Along this line, the operator reduces to u_{yy}.

If the roots of the equation are equal throughout a region, then the operator is parabolic throughout the region, and there is a similar reduction in the form of the operator.

Because the roots of the equation are equal, there is only one characteristic variable, ξ say, which can be chosen to eliminate the term $u_{\xi\xi}$ from the new form of the operator.

However, when the roots are equal their value is $X = -h/a$, and furthermore, the coefficients satisfy $h^2 = ab$, so that, if $a \neq 0$, $b = h^2/a$. (If $a = 0$, then $h = 0$ also, and the form is already reduced.)

The characteristic variable ξ satisfies the equation

$$a(x, y)\xi_x + h(x, y)\xi_y = 0$$

The coefficient of $2u_{\xi\eta}$ in the expansion of the operator is

$$\begin{aligned}
&a(x, y)\xi_x\eta_x + h(x, y)(\xi_x\eta_y + \xi_y\eta_x) + b(x, y)\xi_y\eta_y \\
&= \eta_x(a(x, y)\xi_x + h(x, y)\xi_y) + \eta_y(h(x, y)\xi_x + \frac{h^2(x, y)}{a(x, y)}\xi_y) \\
&= \eta_x(0) + \frac{h(x, y)}{a(x, y)}\eta_y(a(x, y)\xi_x + h(x, y)\xi_y) \\
&= 0 \quad \text{irrespective of the choice of } \eta \ .
\end{aligned}$$

Hence we can choose our second variable to suit ourselves, provided only that it is independent of ξ.

For example, consider the operator

$$u_{xx} + 4u_{xy} + 4u_{yy}$$

The quadratic equation $X^2 + 4X + 4 = 0$ has the equal roots $-2, -2$. Therefore this operator is parabolic throughout the plane.

The characteristic variable ξ satisfies

$$\xi_x + 2\xi_y = 0$$
$$\frac{dx}{dt} = 1\ ,\ \frac{dy}{dt} = 2 = 2\frac{dx}{dt}$$
$$2dx - dy = 0\ ,\ 2x - y = c(s) = \xi \text{ say}$$

For η we can choose $\eta = x$ or $\eta = y$ for their simplicity, or possibly $\eta = x + 2y$ if we wish to maintain an orthogonal co-ordinate system.

In any case, $\xi_x = 2$, $\xi_y = -1$, and the second derivatives all vanish, so that the coefficient of $u_{\xi\xi}$ is

$$1(2)^2 + 4(2)(-1) + 4(-1)^2 = 0$$

the coefficient of $u_{\xi\eta}$ is

$$2(1(2)\eta_x + 2(2)\eta_y + 2(-1)\eta_x + 4(-1)\eta_y) = 0$$

and the coefficient of u_ξ is also 0.

If we choose $\eta = x$, the only non-zero partial derivative of η is $\eta_x = 1$, and the form reduces to $u_{\eta\eta}$.

If we choose $\eta = y$ the only non-zero partial derivative of η is $\eta_y = 1$, and the form reduces to $4u_{\eta\eta}$.

Finally, if we choose $\eta = x + 2y$, the form reduces to $25u_{\eta\eta}$.

Parabolic operators can also have variable coefficients. For example

$$y^2 u_{xx} + 2xyu_{xy} + x^2 u_{yy}$$

The quadratic equation $y^2X^2 + 2xyX + x^2 = 0$ has the equal roots $X = -x/y$, and the characteristic variable satisfies

$$y\xi_x + x\xi_y = 0$$
$$\frac{dx}{dt} = y\ ,\ \frac{dy}{dt} = x\ ,\ x\frac{dx}{dt} = y\frac{dy}{dt}$$
$$x\,dx - y\,dy = 0\ ,\ \frac{1}{2}(x^2 - y^2) = c(s) = \xi$$

Again we can choose η arbitrarily. Specifically, I will choose $\eta = xy$, since this provides an orthogonal co-ordinate system. Note however that this is an idiosyncratic choice, and need not be followed by the student.

With these new variables, $\xi_x = x$, $\xi_y = -y$, $\xi_{xx} = 1$, $\xi_{xy} = 0$, $\xi_{yy} = -1$, $\eta_x = y$, $\eta_y = x$, $\eta_{xx} = 0$, $\eta_{xy} = 1$ and $\eta_{yy} = 0$.

Substituting these expressions into the form for the expansion, we obtain

$$\begin{aligned} & u_{\xi\xi}(y^2x^2 + 2xy(-xy) + x^2(-y)^2) \\ & +2u_{\xi\eta}(y^2(xy) + xy(x^2 - y^2) + x^2(-xy)) \\ & +u_{\eta\eta}(y^2y^2 + 2xy(yx) + x^2x^2) \\ & +u_\xi(y^2(1) + 2xy(0) + x^2(-1)) \\ & +u_\eta(y^2(0) + 2xy(1) + x^2(0)) \\ = & (x^4 + 2x^2y^2 + y^4)u_{\eta\eta} - (x^2 - y^2)u_\xi + 2xyu_\eta \\ = & 4(\xi^2 + \eta^2)u_{\eta\eta} - 2\xi u_\xi + 2\eta u_\eta \end{aligned}$$

The Classical Equations.

In the first lecture, three classical equations were mentioned.

The Wave Equation.

The (idealized) motion of a vibrating string satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where u is the displacement, x is the length variable, t is the time variable, and c is the *speed of sound*.

The quadratic equation associated with this equation is $X^2 - c^{-2} = 0$, which has two distinct real solutions $X = 1/c$ and $X = -1/c$. Hence this equation is **hyperbolic**. The canonical variables are $\xi = x - ct$ and $\eta = x + ct$. In terms of these variables, the equation reduces to

$$u_{\xi\eta} = 0$$

and the general solution is $u = f(x - ct) + g(x + ct)$.

Laplace's Equation.

The steady state distributions associated with heat flow and electrostatic fields in two dimensions satisfy the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ .$$

This is an elliptic equation which is already in its canonical form.

Solutions of Laplace's equation are called **harmonic functions**.

The Heat Equation.

The idealised one-dimensional diffusion of heat (in a rod) satisfies the equation

$$\kappa\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

u represents the temperature, x the spatial variable, t the time variable, and κ is the diffusivity.

This equation is parabolic.

Extensions.

This classification system can be extended to second order linear operators in more than two variables. However, the concept of characteristic variables cannot be generalised.

For the general second order linear operator

$$\sum_{i=1}^{n} a_{ii}(\underline{x}) \frac{\partial^2 u}{\partial x_i^2} \; + \; 2 \sum_{i=2}^{n} \sum_{j=1}^{i-1} a_{ij}(\underline{x}) \frac{\partial^2 u}{\partial x_i \partial x_j}$$

consider the quadratic form $\underline{\xi}' A \underline{\xi}$, where A is the symmetric matrix whose coefficients are the functions a_{ij}.

The operator is classified as **elliptic** wherever the form is either positive definite or negative definite, as **hyperbolic** wherever the form has full rank but is indefinite, and as **parabolic** otherwise. This classification agrees with that adopted for the operators with two independent variables.

Specificly, the following equations are commonly met.

Elliptic equations.

The Laplacean operator can be extended to any number of dimensions. It is commonly designated ∇^2, where

$$\nabla^2 u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

The quadratic form associated with this operator is

$$\xi_1^2 + \xi_2^2 + \; \ldots \; + \xi_n^2$$

which is obviously positive definite. (Its value is positive except when $\xi_1 = \xi_2 = \cdots = \xi_n = 0$.)

In particular we have Laplace's equation in three dimensions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \; .$$

Hyperbolic equations.

The wave equation can be extended to more than one dimension.

In two dimensions we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

In three dimensions it becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

Note that the left-hand sides of these equations are Laplacean operators.

The quadratic form associated with the second of these equations is

$$\xi_1^2 + \xi_2^2 + \xi_3^2 - c^{-2} \xi_4^2$$

which can take both positive and negative values, and is therefore indefinite. However, its rank is four, which is equal to the number of independent variables.

Parabolic equations.

In a similar fashion the Heat equation can be extended to higher dimensions.

In two dimensions it becomes

$$\kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial u}{\partial t}$$

and in three dimensions

$$\kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \frac{\partial u}{\partial t}$$

Once again the left-hand sides of these equations are Laplacean.

In this case the quadratic form associated with the second equation is

$$\xi_1^2 + \xi_2^2 + \xi_3^2 + 0 \times \xi_4^2$$

where the last term has been included to highlight the point that there is one variable missing from the form, which has rank $3 < 4$.