

MATH 3403

WEEK 10

Half range Fourier Transforms.

Just as the normal Fourier series expansion can be generalised to the Fourier Integral Transform, so the half range sine and cosine series can be generalised to give the Fourier Sine Transform and the Fourier Cosine Transform.

The Fourier Sine Transform is defined by the transform pair

$$\begin{aligned}\mathcal{F}_S(\omega) &= \int_0^\infty \sin(\omega x) f(x) dx \\ f(x) &= \frac{2}{\pi} \int_0^\infty \sin(\omega x) \mathcal{F}_S(\omega) d\omega\end{aligned}$$

provided that the integrals exist. In particular, we require that the functions tend to zero as $x \rightarrow \infty$.

Note that, apart from the factor $2/\pi$, the inversion formula has the same form as the transform itself. Some authors convert this into an identity by having a factor $\sqrt{2/\pi}$ in front of each integral.

For example, consider

$$f(x) = \begin{cases} 1 & 0 \leq x \leq a \\ 0 & x > a \end{cases}$$

The Fourier Sine Transform of f is

$$\begin{aligned}\mathcal{F}_S(\omega) &= \int_0^a \sin(\omega x) dx \\ &= -\frac{1}{\omega} \cos(\omega x) \Big|_0^a \\ &= \frac{1 - \cos(\omega a)}{\omega}\end{aligned}$$

From the inversion formula it follows that

$$\int_0^\infty \sin(\omega x) \frac{1 - \cos(a\omega)}{\omega} d\omega = \frac{\pi}{2} f(x)$$

which, by changing the roles of x and ω gives the transform

$$\int_0^\infty \sin(\omega x) \frac{1 - \cos(ax)}{x} dx = \begin{cases} \frac{\pi}{2} & 0 \leq \omega \leq a \\ 0 & \omega > a \end{cases}$$

Similarly, the Fourier Cosine Transform is defined by the transform pair

$$\begin{aligned}\mathcal{F}_C(\omega) &= \int_0^{\infty} \cos(\omega x) f(x) dx \\ f(x) &= \frac{2}{\pi} \int_0^{\infty} \cos(\omega x) \mathcal{F}_C(\omega) d\omega\end{aligned}$$

Taking the same function $f(x)$ as above, we have

$$\begin{aligned}\mathcal{F}_C(\omega) &= \int_0^a \cos(\omega x) dx \\ &= \frac{1}{\omega} \sin(\omega x) \Big|_0^a \\ &= \frac{\sin(\omega a)}{\omega}\end{aligned}$$

so that, from the inversion formula we can obtain

$$\int_0^{\infty} \cos(\omega x) \frac{\sin(ax)}{x} dx = \begin{cases} \frac{\pi}{2} & 0 < \omega < a \\ 0 & \omega > a \end{cases}$$

These transforms are used for semi-infinite problems where the derivatives are of even order. Specifically, the Fourier Sine Transform is used when the boundary condition is of Dirichlet's form and the Fourier Cosine Transform is used when the boundary condition is of Neumann's form.

If $\mathcal{F}_S(\omega)$ is the Fourier Sine Transform of $f(x)$, then the Fourier Sine Transform of $f''(x)$ is given by

$$\begin{aligned}\int_0^{\infty} \sin(\omega x) f''(x) dx &= \sin(\omega x) f'(x) \Big|_0^{\infty} - \omega \int_0^{\infty} \cos(\omega x) f'(x) dx \\ &= -\omega \cos(\omega x) f(x) \Big|_0^{\infty} - \omega^2 \int_0^{\infty} \sin(\omega x) f(x) dx \\ &= \omega f(0) - \omega^2 \mathcal{F}_S(\omega)\end{aligned}$$

Note the difference between this transform and the Laplace Transform. In order to find the Laplace Transform of the second derivative, it is necessary to know both $f(0)$ and $f'(0)$. Hence the Laplace Transform is suited to initial value problems. In contrast, the Fourier Sine Transform of the second derivative requires only knowledge of $f(0)$, so that it is suited to Dirichlet problems.

Consider the following semi-infinite heat equation problem:

$$\begin{aligned}\frac{\partial u}{\partial t} &= K \frac{\partial^2 u}{\partial x^2} \\ 0 < x < \infty \quad 0 < t < \infty \\ u(x, 0) &= 0 \quad u(0, t) = 1\end{aligned}$$

If $\mathcal{U}(\omega, t)$ is the Fourier Sine Transform of $u(x, t)$;

$$\mathcal{U}(\omega, t) = \int_0^{\infty} \sin(\omega x) u(x, t) dx ;$$

then

$$\int_0^{\infty} \sin(\omega x) u_{xx}(x, t) dx = \omega u(0, t) - \omega^2 \mathcal{U} = \omega - \omega^2 \mathcal{U} .$$

Applying the transform to the equation and the initial conditions we have

$$\begin{aligned} \frac{\partial \mathcal{U}}{\partial t} &= K(\omega - \omega^2 \mathcal{U}) \\ \mathcal{U}(\omega, 0) &= \int_0^{\infty} \sin(\omega x) u(x, 0) dx = 0 \end{aligned}$$

This is a first order linear ordinary differential equation with constant coefficients for \mathcal{U} as a function of t , whose solution is

$$\mathcal{U}(\omega, t) = \frac{1}{\omega} \left(1 - e^{-K\omega^2 t} \right) .$$

Using the inverse transform formula, we obtain the solution of the original problem in integral form as

$$u(x, t) = \frac{2}{\pi} \int_0^{\infty} \sin(\omega x) \frac{1}{\omega} \left(1 - e^{-K\omega^2 t} \right) d\omega$$

which can be expressed in terms of the error function

$$\operatorname{erf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-y^2} dy$$

as

$$u(x, t) = 1 - \operatorname{erf}(x/2\sqrt{Kt}) = \operatorname{erfc}(x/2\sqrt{Kt}) .$$

Similarly, the Fourier Cosine Transform of $f''(x)$ is given by

$$\begin{aligned} \int_0^{\infty} \cos(\omega x) f''(x) dx &= \cos(\omega x) f'(x) \Big|_0^{\infty} + \omega \int_0^{\infty} \sin(\omega x) f'(x) dx \\ &= -f'(0) + \omega \sin(\omega x) f(x) \Big|_0^{\infty} - \omega^2 \int_0^{\infty} \cos(\omega x) f(x) dx \\ &= -f'(0) - \omega^2 \mathcal{F}_C(\omega) \end{aligned}$$

Since this transform requires only knowledge of $f'(0)$ it is suited to Neumann problems.

For example, consider the Wave equation

$$\begin{aligned} u_{tt} &= u_{xx} ; t > 0 , x > 0 \\ u(x, 0) &= 0 , u_t(x, 0) = 0 , u_x(0, t) = 1 \end{aligned}$$

If $\mathcal{U}(\omega, t)$ is the Fourier Cosine Transform of $u(x, t)$;

$$\mathcal{U}(\omega, t) = \int_0^{\infty} \cos(\omega x) u(x, t) dx ;$$

then

$$\int_0^{\infty} \cos(\omega x) u_{xx}(x, t) dx = -u_x(0, t) - \omega^2 \mathcal{U} = -1 - \omega^2 \mathcal{U} .$$

Applying this transform to the equation and initial condition we have the second order linear equation

$$\begin{aligned} \frac{\partial^2 \mathcal{U}}{\partial t^2} &= -\omega^2 \mathcal{U} - 1 \\ \mathcal{U}(\omega, 0) &= \int_0^{\infty} \cos(\omega x) u(x, 0) dx = 0 \\ \mathcal{U}_t(\omega, 0) &= \int_0^{\infty} \cos(\omega x) u_t(x, 0) dx = 0 \end{aligned}$$

whose solution is

$$\mathcal{U}(\omega, t) = \frac{\cos(\omega t) - 1}{\omega^2}$$

Hence

$$u(x, t) = \frac{2}{\pi} \int_0^{\infty} \cos(\omega x) \frac{\cos(\omega t) - 1}{\omega^2} d\omega = \begin{cases} x - t & 0 \leq x \leq t \\ 0 & t > x \end{cases}$$

Finally, consider the following elliptic problem:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 ; 0 < x < \infty ; 0 < y < \infty ; \\ u_x(0, y) &= 0 ; u(x, 0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases} \\ u(x, y) &\rightarrow 0 \text{ as } |x| + |y| \rightarrow \infty \end{aligned}$$

We have Dirichlet boundary conditions on the line $y = 0$, therefore we can employ a Fourier Sine Transform with respect to the variable y .

We have Neumann boundary conditions on the line $x = 0$, so that we can employ a Fourier Cosine Transform with respect to the variable x .

Let

$$\mathcal{V}(\omega, y) = \int_0^{\infty} \cos(\omega x) u(x, y) dx .$$

Then

$$\begin{aligned} -\omega^2 \mathcal{V} + \mathcal{V}_{yy} &= 0 \\ \mathcal{V}(\omega, 0) &= \int_0^1 \cos(\omega x) dx = \frac{\sin(\omega)}{\omega} \end{aligned}$$

Let

$$\mathcal{U}(\omega, \alpha) = \int_0^{\infty} \sin(\alpha y) \mathcal{V}(\omega, y) dy .$$

Then

$$\begin{aligned}
 -\omega^2 \mathcal{U} - \alpha^2 \mathcal{U} + \alpha \frac{\sin(\omega)}{\omega} &= 0 \\
 \mathcal{U} &= \frac{\alpha}{\alpha^2 + \omega^2} \frac{\sin(\omega)}{\omega} \\
 \mathcal{V} &= \frac{2}{\pi} \frac{\sin(\omega)}{\omega} \int_0^\infty \frac{\alpha}{\alpha^2 + \omega^2} \sin(\alpha y) d\alpha \\
 \mathcal{V}(\omega, y) &= \frac{\sin(\omega)}{\omega} e^{-\omega y} \quad (\text{from the tables}) \\
 u(x, y) &= \frac{2}{\pi} \int_0^\infty \frac{\sin(\omega) \cos(\omega x)}{\omega} e^{-\omega y} d\omega \\
 &= \frac{1}{\pi} \int_0^\infty \frac{\sin(\omega(x+1))}{\omega} e^{-\omega y} - \frac{1}{\pi} \int_0^\infty \frac{\sin(\omega(x-1))}{\omega} e^{-\omega y}
 \end{aligned}$$

These final integrals have the form of Laplace Transform integrals.
If

$$\begin{aligned}
 F(a) &= \int_0^\infty \frac{\sin(a\omega)}{\omega} e^{-\omega y} d\omega \\
 F'(a) &= \int_0^\infty \cos(a\omega) e^{-\omega y} d\omega \\
 &= \frac{y}{a^2 + y^2} \\
 F(0) &= 0 \\
 F(a) &= \int_0^a \frac{y}{s^2 + y^2} ds = \arctan \left(\frac{a}{y} \right)
 \end{aligned}$$

Therefore

$$u(x, y) = \frac{1}{\pi} \left(\arctan \left(\frac{x+1}{y} \right) - \arctan \left(\frac{x-1}{y} \right) \right) = \frac{\theta}{\pi}$$

where θ is the angle subtended at (x, y) by the line segment between $(-1, 0)$ and $(1, 0)$.