MATH 3403
WEEK 1

1. Definitions and examples.

A partial differential equation is an expression relating an unknown function of more than one variable with one or more of its derivatives.

For example

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 1$$

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \quad \text{The Heat equation}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{The Wave equation}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y) \quad \text{Poisson's equation}$$

The **order** of the equation is the order of the highest derivative present. In the above examples, the first two equations are *first order*, while the remaining three are *second order*.

An equation is **linear** if none of the terms in the equation involves the product of the function and or any of its derivatives.

Of the equations above, all but the second are *linear*. Since the second equation involves the products of the first partial derivatives with themselves, this equation is *non-linear*.

When every term of a linear equation involves the unknown function or one of its derivatives, the equation is said to be **homogeneous**.

Of the above equations, the first, third and fourth are *homogeneous linear equations*. The Poisson equation is only *homogeneous* if $F(x,y) \equiv 0$. In this case, it is referred to as **Laplace's equation**.

Homogeneous equations are of particular interest because their solutions are additive; that is

(a) if u_1 and u_2 are solutions of a linear homogeneous equation, so is $Au_1 + Bu_2$ for every pair of constants A and B.

(b) if $u(\underline{x}, \alpha)$ is a solution of a linear homogeneous equation for every value of the parameter α in some range $a \le \alpha \le b$, then, subject to convergence requirements, so is $\int_a^b f(\alpha)u(\underline{x}, \alpha)\, d\alpha$.

For example,

$$u(x,t,\lambda) = e^{-\lambda^2 \kappa t}\sin(\lambda x)$$

is a solution of the heat equation for all values of λ.

Check:

$$\frac{\partial u}{\partial x} = \lambda e^{-\lambda^2 \kappa t} \cos(\lambda x)$$
$$\frac{\partial^2 u}{\partial x^2} = -\lambda^2 e^{-\lambda^2 \kappa t} \sin(\lambda x)$$
$$\frac{\partial u}{\partial t} = -\lambda^2 \kappa e^{-\lambda^2 \kappa t} \sin(\lambda x) = \kappa \frac{\partial^2 u}{\partial x^2}$$

Therefore, for suitable choices of $f(\lambda)$

$$\int_0^\infty f(\lambda) e^{-\lambda^2 \kappa t} \sin(\lambda x)\, d\lambda$$

is also a solution of the heat equation[1] , and for suitable choices of the coefficients $a_1, a_2, \ldots$ the infinite sum

$$\sum_{n=1}^{\infty} a_n e^{-n^2 \kappa t} \sin(nx)$$

is a solution[2] which is zero when $x = 0$ and $x = \pi$. (We will return to these example at a later date.)

Change of variables.

When we choose our initial independent variables to derive our partial differential equation, there is no guarantee that they are the optimum variables with which to derive the solution.

For example, if we consider the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

we find that it is easier to solve if we make the change of variable

$$\xi = x + ct \quad \eta = x - ct$$
$$x = \frac{1}{2}(\xi + \eta) \quad t = \frac{1}{2c}(\xi - \eta)$$

Using the Chain Rule,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$
$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = c\frac{\partial u}{\partial \xi} - c\frac{\partial u}{\partial \eta}$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$
$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial \xi^2} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c^2\frac{\partial^2 u}{\partial \eta^2}$$

[1] This form of solution is known as a Fourier Sine Transform, of which more anon.
[2] This is a Fourier Sine Series.

Substituting these expressions into the wave equation we obtain

$$4\frac{\partial^2 u}{\partial\xi\partial\eta} = 0$$

in which form it is easy to obtain a general solution

$$u = F(\xi) + G(\eta)$$

Note that this solution involves two arbitrary **functions** F and G, whereas the general solution of a second order *ordinary* differential equation involves two arbitrary constants.

General solution.

If we can find a solution of an n-th order partial differential equation involving n independent arbitrary functions, then we call such an expression a **general solution** of the equation.

For example, $u = f(y - x^2)$ is a general solution of the first order equation

$$\frac{\partial u}{\partial x} + 2x\frac{\partial u}{\partial y} = 0$$

Check: setting $\xi = y - x^2$,

$$\begin{aligned} \frac{\partial u}{\partial x} &= f'(\xi)\frac{\partial \xi}{\partial x} = -2xf'(\xi) \\ \frac{\partial u}{\partial y} &= f'(\xi)\frac{\partial \xi}{\partial y} = f'(\xi) \\ \frac{\partial u}{\partial x} &+ 2x\frac{\partial u}{\partial y} = 0 \end{aligned}$$

First order linear equations.

The most general first order linear equation in two variables is

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y)u + f(x,y)$$

In order to discover the structure of the solutions of this equation, we start by considering the simplest possible case:

$$\frac{\partial u}{\partial y} = 0$$

whose general solution is $u = F(x)$. In other words, the solution is a constant along the lines x constant. To specify a particular solution of this equation, we need to specify the value of u once and only once for each value of x in our range of interest, and when we have done so, we have specified the solution throughout the strip $-\infty < y < \infty$.

Initial values.

When a particular solution of a first order partial differential equation is required, it is usual to specify **initial values** along some curve C in the $x-y$ plane. Specifically, if we parametrise C by $\{x(s), y(s)\}$, $\alpha \le s \le \beta$, then we specify the value of u as $u(s)$. Where the initial values are given in non-parametric form, it is usual to rewrite them in parametric form. For example, the non-parametric initial values

$$u(x, 0) = e^x$$

can be expressed patametrically as

$$u = e^s \text{ on the line } x = s, y = 0 \;\; -\infty < s < \infty$$

Returning to our simple equation, we see that we can solve the initial value problem

$$\frac{\partial u}{\partial y} = 0 \;,\;\; u = u(s) \text{ on } C$$

provided that none of the lines x constant intersect C in more than one place; that is, provided $x(s_1) \ne x(s_2)$ for $\alpha \le s_1 < s_2 \le \beta$. Otherwise, we have $u(x(s_1)) = u(s_1) = u(s_2)$ which is not usually true. The lines x constant in the $x-y$ plane are called the **characteristics** of the equation. In addition, the solution is defined only at those points which can be reached from the initial line along a characteristic.

Note that the value of the solution at any point of the characteristic depends only on the value of the solution at the point where the characteristic intersects the initial line, and that if the function $u(s)$ has a discontinuity at $s = s_0$, then the solution is discontinuous along the characteristic $x = x(s_0)$.

If we progress to the slightly more general problem

$$\frac{\partial u}{\partial y} = f(x, y) \;,\;\; u = u(s) \text{ on } x = x(s), y = y(s), \alpha \le s \le \beta$$

we can obtain the solution for each value of s directly by integrating, in the form

$$u(x(s), y) = u(s) + \int_{y(s)}^{y} f(x(s), t)\, dt$$

provided again that different values of s determine different values of x.

In common with the first example, the value of the solution at any point of a characteristic is determined solely by the value of u at the point of intersection of the the characteristic x constant with the initial line, the solution is defined at a point (x, y) only if this point can be reached from the initial line along a contour, and if $u(s)$ is discontinuous at $s = s_0$, so is the solution along the characteristic $x = x(s_0)$.

For example: Find the solution of

$$\frac{\partial u}{\partial y} = x + y \;;\;\; u(x, x) = e^x$$

We begin by parametrising the initial data. If we set $x = s$, then on the initial line, $y = s$ and $u = e^s$.

For fixed s, we have the ordinary differential equation

$$\frac{du}{dy} = s + y \; ; u = e^s \text{ when } y = s$$

whose solution is

$$u = e^s + s(y - s) + \frac{1}{2}(y^2 - s^2)$$

Substituting $s = x$ into this expression gives the solution

$$u(x, y) = e^x + \frac{1}{2}y^2 + xy - \frac{3}{2}x^2$$

The general first order linear equation.

In order to solve the general first order linear equation, we look for a change of variable which will reduce the equation to a form similar to the cases which we have just considered.

If the expression

$$a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y}$$

reduces to the form $\frac{\partial u}{\partial t}$ then the functions x and y in terms of t are related by the chain rule

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

and the first order partial differential equation can be replaced by the system of first order ordinary differential equations

$$\begin{aligned} \frac{dx}{dt} &= a(x, y) \\ \frac{dy}{dt} &= b(x, y) \\ \frac{du}{dt} &= c(x, y)u + f(x, y) \end{aligned}$$

These equations are known as the **characteristic equations** associated with the partial differential equation.

The first two equations depend solely on the left hand side (i.e. the terms involving the derivatives) of the pde. They are a second order autonomous system, whose solutions can be represented as trajectories in the phase plane. These trajectories are the **characteristic field** of the equation.

If we solve the equations for initial data $\{x(s), y(s), u(s)\}$, then from the first two equations we obtain solutions $x(t, s)$ and $y(t, s)$ which we can substitute into the third equation to obtain a first order linear ordinary differential equation for u as a function of t, whose solution will be of the form $u(t, s)$.

In order to express u in terms of x and y, it is necessary to invert the relations $x = x(t, s)$ and $y = y(t, s)$ in order to obtain $t = t(x, y)$ and $s = s(x, y)$. From the *Inverse Function Theorem*, we know that this is possible provided that the Jacobian is not zero. The Jacobian is

$$\frac{\partial(x, y)}{\partial(t, s)} = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{vmatrix}$$

which vanishes when $(\partial x/\partial t, \partial y/\partial t)$ is a multiple of $(\partial x/\partial s, \partial y/\partial s)$. The first of these vectors is tangent to the characteristic field, and the second is tangent to the initial line. If they are multiples one of another, then these fields are parallel at some point. Therefore the requirement that the functions be invertible is equivalent to the requirement that the initial line be transverse to the characteristic field, which we have already noted in the simple case considered above.

For example, consider the following initial value problem

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u \ , \ u(x,0) = f(x) \ .$$

Firstly, we parametrise the initial data.

If we set $x = s$, then $y = 0$ and $u = f(s)$ on the initial line.

The characteristic equations for this problem are

$$\begin{aligned} \frac{dx}{dt} &= 1 \ , \ x = s \text{ when } t = 0 \\ \frac{dy}{dt} &= 1 \ , \ y = 0 \text{ when } t = 0 \\ \frac{du}{dt} &= u \ , \ u = f(s) \text{ when } t = 0 \end{aligned}$$

(Since the system is autonomous, we can choose the initial time to suit ourselves. It is usually convenient to set $t = 0$ on the initial line.)

The first two equations have the simple solutions $x = t + s$, $y = t$. Eliminating t, we obtain $x - y = s$, which shows that the characteristic field consists of parallel lines inclined at an angle of 45^o to the x-axis. This field is never parallel to the initial line (the x-axis), so there are no impediments to the solution.

Solving the third equation we obtain $u = f(s)e^t$. Since $s = x - y$ and $t = y$, this becomes

$$u(x,y) = f(x-y)e^y \ .$$

By setting u as an arbitrary function on the initial line, we have in fact derived a general solution for the original partial differential equation.

Characteristic equations.

The solution of the partial differential equation

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + f(x,y)$$

leads to the consideration of the system

$$\begin{aligned} \frac{dx}{dt} &= a(x,y) \\ \frac{dy}{dt} &= b(x,y) \\ \frac{du}{dt} &= c(x,y)u + f(x,y) \end{aligned}$$

of *characteristic equations*.

When a and b are constants, as in the previous exercise, the characteristic field satisfies

$$\frac{dy}{dx} = \frac{b}{a} \ , \ y = \frac{b}{a}x + \text{ const}$$

which represents a family of parallel lines covering the whole plane. In this case, if the initial data is given on an infinite transverse curve, the solution is defined throughout the $x - y$ plane.

However, when a and b depend on x and y, the characteristic field is no longer a system of parallel straight lines. Consequently, in most cases the characteristic field and the initial curve become parallel, which places restrictions on the initial data and the solution. In addition, the equations may have singular points if $a(x, y)$ and $b(x, y)$ vanish simultaneously.

When the initial line is parallel to the characteristic field at some point, there are restrictions on the amount of data that we can prescribe, and the solution may only be defined in part of the plane.

Consider for example the initial value problem

$$u_x + 2xu_y = 0 \ , \ u = f(s) \text{ when } x = as, y = bs, (a, b) \neq (0, 0)$$

The characteristic field for this equation is given by solving

$$\frac{dx}{dt} = 1 \ , \ \frac{dy}{dt} = 2x \ , \ \frac{dy}{dx} = 2x \ , \ y = x^2 + c$$

which gives a family of parabolas. The initial line is specified as a straight line through the origin, and unless $a = 0$, that is the initial line is the y-axis, this line will be tangent to the characteristic field at some point and the problem is ill-posed.

Suppose that we naively attempt to find the solution anyway.

Solving the characteristic equations we obtain

$$x = t + as \ , \ y = t^2 + 2ast + bs \ , u = f(s)$$

and when we attempt to determine s in terms of x and y we are led to the equation

$$y = x^2 - a^2 s^2 + bs$$

which has two solutions for s, $(b \pm \sqrt{b^2 + 4a^2(x^2 - y)})/2a^2$, unless $a = 0$. This means that the characteristic through (x, y) intersects the initial line in two points if the discriminant $b^2 + 4a^2(x^2 - y)$ is positive, both of which attempt to specify the solution, leading to a contradiction. Where the discriminant is negative, the solution is not defined, since no characteristic through this point intersects the initial line.

These regions are bounded by the parabola which is tangent to the initial line.

When $a = 0$, s is uniquely determined by (x, y), and the solution is $u = f((y - x^2)/b)$.

For other values of a, we can obtain a solution to the initial value problem only by restricting the initial data to the half-line one or other side of the point of tangency. For example, if we take $a = 1$, $b = 0$, so that the initial line is the x-axis, then we can ensure a solution if we restrict ourselves to $s > 0$, which means that the initial data $u = f(s)$ is only specified on the half axis $x > 0$, and the solution $u = f(\sqrt{x^2 - y})$ in turn is only defined in the region below the parabola $y = x^2$.

Note that in this case the restriction $s > 0$ means that only one solution of the equation $y = x^2 - s^2$ is valid.

Singular points in the characteristic field bring their own peculiarities.

For example, consider the following initial value problem.

$$xu_x + yu_y = u \ , \ u(1, y) = y^2$$

Here the initial data is given on the infinite line $x = 1$.

However, the characteristic field is given by solving the autonomous system

$$\frac{dx}{dt} = x \ , \ \frac{dy}{dt} = y$$

which has a singular point — a **proper node** or **star node** — at the origin.

$$\{ \quad \begin{array}{l} \frac{dx}{dt} = x \ , \ x = ae^t \\ \frac{dy}{dt} = y \ , \ y = be^t \\ ay = bx \qquad \} \end{array}$$

The characteristic field consists of rays emanating from the origin, and the solution is defined in that part of the plane which can be reached from the initial line along characteristics, namely the right half-plane $x > 0$. This region is bounded by the characteristic rays $x = 0, y > 0$, and $x = 0, y < 0$. Note that these characteristics are parallel to the initial line; the condition which leads to the vanishing of the Jacobian and consequent non-invertability.

In order to find the required solution, we first parametrise the initial data as $x = 1$, $y = s$, $u = s^2$.

The solution follows by solving

$$\begin{array}{l} \frac{dx}{dt} = x \ , \ x(0) = 1 \ , \ x = e^t \\ \frac{dy}{dt} = y \ , \ y(0) = s \ , \ y = se^t \end{array}$$

Inverting these relations, we obtain $s = y/x$ and $t = \log x$, both of which are undefined when $x = 0$.

The solution for u follows by solving

$$\frac{du}{dt} = u \ , \ u(0) = s^2 \ , \ u = s^2 e^t$$

and substituting for s and t to obtain $u = y^2/x$.

If we change the initial data so that $u = 1$ when $|s| < 1$, and $u = 0$ otherwise, then all we need to change in the solution is the initial value of u. Doing so, we get

$$u = e^t \ , \ |s| < 1 \ , \ u = 0 \ , \ \text{otherwise}$$

Substituting for s and t, we get

$$u = x \ , x > |y| \ , \ u = 0 \ , \ 0 < x \leq |y|$$

Note that the discontinuities in the initial data propogate along the characteristics.

Another example: Solve the initial value problem

$$(x^2+y^2)u_x + 2xyu_y = (x+y)u \ ; \ u = y^2 \text{ when } x = 0 \ , \ y > 0$$

The initial data in parametric form is $x = 0$, $y = s$, $u = s^2$, $s > 0$.

The characteristic field is found by solving the system

$$\frac{dx}{dt} = x^2 + y^2$$
$$\frac{dy}{dt} = 2xy$$

This system has a second order singular point at the origin.

Eliminating dt between the equations we obtain

$$2xy\frac{dx}{dy} = x^2 + y^2$$

which is a linear equation in $X = x^2$; viz

$$y\frac{dX}{dy} - X = y^2$$
$$\frac{d}{dy}\left(\frac{X}{y}\right) = 1$$
$$X = x^2 = y^2 + cy$$

Therefore the characteristic field is a collection of hyperbolas, which intersect the y-axis when $y = 0$ and $y = -c$, and which are asymptotic to the lines $x = \pm(y + \frac{1}{2}c)$.

With respect to the initial value problem under discussion, if $x = 0$ when $y = s$, then $c = -s$, where $s > 0$, so that we need only consider the upper branches of the parabolas given by $y = \frac{s}{2} + \sqrt{x^2 + \frac{1}{4}s^2}$. The lower branches all intersect the y-axis at the origin, which is not part of the initial line.

In order to express x and y in terms of t we need to eliminate one or other of them from the appropriate equation and solve the resulting first order equation. Using the expression derived above for y, the first equation reduces to

$$\begin{aligned}\frac{dx}{dt} &= x^2 + \left(\frac{s}{2} + \sqrt{x^2 + \frac{1}{4}s^2}\right)^2 \\ &= x^2 + \frac{1}{4}s^2 + s\sqrt{x^2 + \frac{1}{4}s^2} + x^2 + \frac{1}{4}s^2 \\ &= 2\left(x^2 + \frac{1}{4}s^2\right) + s\sqrt{x^2 + \frac{1}{4}s^2}\end{aligned}$$

Setting $x = \frac{s}{2}\sinh\theta$, this equation becomes

$$\frac{s}{2}\cosh\theta\frac{d\theta}{dt} = 2\left(\frac{1}{4}s^2\cosh^2\theta\right) + \frac{1}{2}s^2\cosh\theta$$
$$\frac{d\theta}{dt} = s(1+\cosh\theta) = 2s\cosh^2\left(\frac{1}{2}\theta\right)$$
$$\frac{1}{2}\text{sech}^2\left(\frac{1}{2}\theta\right)d\theta = s\,dt$$
$$\tanh\left(\frac{1}{2}\theta\right) = st$$

where we have used the fact that when $t = 0$, $x = 0$, and therefore $\theta = 0$.

Using the hyperbolic equivalent of the half angle formulae, we obtain

$$x = \frac{s}{2}\sinh\theta = \frac{s}{2}\frac{2st}{1-s^2t^2} = \frac{s^2t}{1-s^2t^2}$$

and

$$\begin{aligned} y &= \frac{s}{2} + \sqrt{x^2 + \frac{1}{4}s^2} \\ &= \frac{s}{2} + \sqrt{\frac{1}{4}s^2(\sinh^2\theta + 1)} \\ &= \frac{s}{2}(1+\cosh\theta) \\ &= s\cosh^2\left(\frac{1}{2}\theta\right) \\ &= \frac{s}{1-s^2t^2} \end{aligned}$$

By way of a check on the calculations we can calculate

$$\frac{dy}{dt} = \frac{2s^3t}{(1-s^2t^2)^2} = 2xy$$

We can now express the differential equation for u in terms of t;

$$\frac{du}{dt} = \frac{s+s^2t}{1-s^2t^2}u \ = \ \frac{su}{1-st} \ , \ u(0) = s^2$$

whose solution is

$$u = \frac{s^2}{1-st}$$

Finally, from the parametric forms for x and y, we get

$$\begin{aligned} st &= x/y \\ s &= y\left(1 - \frac{x^2}{y^2}\right) \\ u &= y^2\left(1-\frac{x^2}{y^2}\right)^2 \frac{1}{1-\frac{x}{y}} \\ u &= \frac{1}{y}(y+x)(y^2-x^2) = y^2 - x^2 + xy - \frac{x^3}{y} \end{aligned}$$

This solution holds for $y > |x|$.