Rayleigh-Ritz Example

Note: The Beta Function is defined by

$$B\,(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

We wish to determine approximately the principal eigenvalue for the equation

$$u'' + \lambda u = 0\ ;\ u(0) = u(1) = 0\ .$$

We consider the class of functions $u = x^a(1-x)^a$.

$$u' = ax^{a-1}(1-x)^a - ax^a(1-x)^{a-1}$$

$$\int_0^1 (u')^2\,dx = a^2\left[\int_0^1 x^{2a-2}(1-x)^{2a}\,dx\right.$$

$$\left. - 2\int_0^1 x^{2a-1}(1-x)^{2a-1}\,dx + \int_0^1 x^{2a}(1-x)^{2a-2}\,dx\right]$$

$$= a^2\left[\frac{\Gamma(2a-1)\Gamma(2a+1)}{\Gamma(4a)} - 2\frac{\Gamma(2a)\Gamma(2a)}{\Gamma(4a)} + \frac{\Gamma(2a+1)\Gamma(2a-1)}{\Gamma(4a)}\right]$$

$$= a^2\frac{\Gamma^2(2a-1)}{\Gamma(4a)}\left[2a(2a-1) - 2(2a-1)^2 + 2a(2a-1)\right]$$

$$= a^2\frac{\Gamma^2(2a-1)}{\Gamma(4a)}2(2a-1)$$

$$\int_0^1 u^2\,dx = \int_0^1 x^{2a}(1-x)^{2a}\,dx$$

$$= \frac{\Gamma^2(2a+1)}{\Gamma(4a+2)} = \frac{\Gamma^2(2a-1)}{\Gamma(4a)}\frac{4a^2(2a-1)^2}{4a(4a+1)}$$

The Rayleigh quotient is

$$R = \frac{2a^2(2a-1)4a(4a+1)}{4a^2(2a-1)^2} = \frac{8a^2+2a}{2a-1)} = 4a + 3 + \frac{3}{2a-1}$$

$$\frac{dR}{da} = 4 - \frac{6}{(2a-1)^2}$$

which vanishes when

$$(2a-1)^2 = \frac{6}{4}\ ;\ 2a-1 = \frac{\sqrt{6}}{2}\ ; 4a = 2+\sqrt{6}$$

The corresponding value of R is

$$R = 2 + \sqrt{6} + 3 + \sqrt{6} = 5 + 2\sqrt{6} = 9.899$$

$$\sqrt{R} = \sqrt{3} + \sqrt{2} = 3.146$$

which compares favorably with the exact value

$$\pi^2 = 9.870\ .$$