MATH 3403

Quadratic and Hermitian Forms

Let A be a real symmetric matrix, and $\underline{x}$ a vector in $\mathbb{R}^n$. Then

$$\underline{x}' A \underline{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i a_{ij} x_j$$

is a (real) quadratic form.

Let H be a complex hermitian matrix, and $\underline{x}$ a vector in $\mathbb{C}^n$. Then

$$\underline{x}^* H \underline{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{x}_i h_{ij} x_j$$

is a (complex quadratic) hermitian form.

In both cases, the form is a real number.

If the form is positive for every non-zero vector, it is said to be **positive definite**.

If the form is never negative, but may vanish for some non-zero vectors, it is said to be **positive semi-definite**.

If the form takes both positive and negative values, it is said to be **indefinite**.

If the form is never positive, but may vanish for some non-zero vectors, it is said to be **negative semi-definite**.

If the form is negative for every non-zero vector, it is said to be **negative definite**.

The nature of a form may be determined by repeatedly **completing the squares**.

e.g.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & c \end{pmatrix}$$

$$\begin{aligned}
\underline{x}' A \underline{x} &= x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + cx_3^2 \\
&= x_1^2 + 2x_1(x_2 + x_3) + (x_2 + x_3)^2 \\
&\quad +2x_2^2 + 4x_2x_3 + cx_3^2 - (x_2^2 + 2x_2x_3 + x_3^2) \\
&= (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 + (c-1)x_3^2 \\
&= (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 + x_3^2 + (c-2)x_3^2 \\
&= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 + (c-2)x_3^2
\end{aligned}$$

Therefore:

(a) If $c > 2$, the expression is positive for every $\underline{x} \neq \underline{0}$. This is a positive definite form.

(b) If $c = 2$, the expression is never negative, but vanishes when

$$\begin{aligned} x_1 + x_2 + x_3 =& 0 \\ x_2 + x_3 =& 0 \\ (x_1, x_2, x_3) =& a(0, 1, -1) \end{aligned}$$

This form is positive semi-definite.

(c) If $c < 2$, the expression takes both positive and negative values (for example for $(1, 0, 0)$ and $(0, 1, -1)$) and the form is indefinite.

Eigenvalues and definiteness

If λ is a (necessarily real) eigenvalue of either A or H, and $\underline{x}$ is the corresponding eigenvector,

$$\underline{x}^* H \underline{x} = \lambda ||\underline{x}||^2$$

Therefore:

(a) If H (A) has both positive and negative eigenvalues the form is indefinite.

(b) If H (A) has a zero eigenvalue there is a non-zero vector for which the form vanishes.

(c) Since the eigenvectors form an (orthogonal) basis, the form is positive definite if and only if all the eiganvalues are positive and positive semi- definite if and only all the eigenvalues are non-negative.

However, for large matrices calculating all the eigenvalues is tedious, and some variant of the 'completing the squares' procedure is usually employed to determine the nature of the form.