MATH 3403

TUTORIAL SHEET 9

SOLUTIONS

1. Let $r^2 = (x-\xi)^2 + (y-\eta)^2$.

On the line $y = 0$, this takes the same value as $\rho^2 = (x-\xi)^2 + (y+\eta)^2$, so that the Green's Function for the upper half plane is

$$G(x, u; \xi, \eta) = \log(\rho) - \log(r)$$
$$= \frac{1}{2}\left[\log\left((x-\xi)^2 + (y+\eta)^2\right) - \log\left((x-\xi)^2 + (y-\eta)^2\right)\right]$$

On the boundary,

$$\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial y}$$
$$= -\frac{1}{2}\left[\frac{2(y+\eta)}{(x-\xi)^2 + (y+\eta)^2} - \frac{2(y-\eta)}{(x-\xi)^2 + (y-\eta)^2}\right]\Bigg|_{y=0}$$
$$= -\frac{2\eta}{(x-\xi)^2 + \eta^2}$$

Hence
$$u(\xi, \eta) = -\frac{1}{2\pi}\int \frac{\partial G}{\partial n} f\, ds = \frac{\eta}{\pi}\int_{-\infty}^{\infty} \frac{f(x)\, dx}{(x-\xi)^2 + \eta^2}\ .$$

2.

$$u(\xi, \eta) = \frac{\eta}{\pi}\int_0^{\infty} \frac{dx}{(x-\xi)^2 + \eta^2}$$
$$= \frac{1}{\pi}\arctan\left(\frac{x-\xi}{\eta}\right)\Bigg|_0^{\infty}$$
$$= \frac{1}{\pi}\left(\frac{\pi}{2} + \arctan\left(\frac{\xi}{\eta}\right)\right)$$
$$= \frac{\pi - \theta}{\pi}$$

where θ is the principal argument of $\zeta = \xi + i\eta$.

3. If $u'' = 1\ ;\ u(0) = u(1) = 0$, then

$$u = \frac{1}{2}x^2 + ax + b$$
$$0 = b\ ;\ 0 = \frac{1}{2} + a$$
$$u = \frac{1}{2}x(x-1)$$

$$
\begin{aligned}
-\frac{1}{2}\int_0^1 G(x,\xi)F(\xi)\,d\xi = & -\frac{c}{2}(1-x)\int_0^x \xi\,d\xi - \frac{c}{2}x\int_x^1 (1-\xi)\,d\xi \\
= & -\frac{c}{2}(1-x)\frac{1}{2}x^2 - \frac{c}{2}x\left(\frac{1}{2} - x + \frac{1}{2}x^2\right) \\
= & -\frac{c}{4}\left(x^2 - x^3 + x - 2x^2 + x^3\right) \\
= & \frac{c}{4}x(x-1)
\end{aligned}
$$

so that $c = 2$.

4.

$$
\begin{aligned}
\int_0^1 G(x,\xi)\sin(n\pi x)\,dx = & 2(1-\xi)\int_0^\xi x\sin(n\pi x)\,dx + 2\xi\int_\xi^1 (1-x)\sin(n\pi x)\,dx \\
= & 2(1-\xi)\left[-\frac{1}{n\pi}x\cos(n\pi x)\Big|_0^\xi + \frac{1}{n\pi}\int_0^\xi \cos(n\pi x)\,dx\right] \\
+ & 2\xi\left[-\frac{1}{n\pi}(1-x)\cos(n\pi x)\Big|_\xi^1 - \frac{1}{n\pi}\int_\xi^1 \cos(n\pi x)\,dx\right] \\
= & 2(1-\xi)\left[-\frac{1}{n\pi}\xi\cos(n\pi\xi) + \frac{1}{n^2\pi^2}\sin(n\pi\xi)\right] \\
+ & 2\xi\left[\frac{1}{n\pi}(1-\xi)\cos(n\pi\xi) + \frac{1}{n^2\pi^2}\sin(n\pi\xi)\right] \\
= & \frac{2}{n^2\pi^2}\sin(n\pi\xi)
\end{aligned}
$$

$$
\begin{aligned}
G(x,\xi) = & \sum_{n=1}^\infty a_n(\xi)\sin(n\pi x) \\
a_n(\xi) = & \int_0^1 G(x,\xi)\sin(n\pi x)\,dx \Big/ \int_0^1 \sin^2(n\pi x)\,dx \\
= & \frac{4}{n^2\pi^2}\sin(n\pi\xi)
\end{aligned}
$$