MATH 3403

Tutorial Sheet 8

Solutions

1. Solve

$$u_{xx} - u_{tt} = x^2; \quad 0 < x < 1 \ ; \quad 0 < t < \infty$$

$$\text{where} \qquad \begin{aligned} u(x,0) &= 0 \ ; \\ u_t(x,0) &= 0 \ ; \\ u(0,t) &= 0 \ ; \\ u_x(1,t) &= 0 \ . \end{aligned}$$

(**Hint:** Find a particular solution of the equation as a function of x which satisfies the boundary conditions.)

Ans: We want a particular solution $U(x)$ which satisfies $U(0) = 0$, $U'(1) = 0$.

$$\begin{aligned} U'' &= x^2 \\ U' &= \frac{1}{3}x^3 + c \\ U'(1) &= \frac{1}{3} + c \ = \ 0 \\ c &= \ -\frac{1}{3} \\ U &= \frac{1}{12}x^4 - \frac{1}{3}x + d \\ U(0) &= d = 0 \\ U(x) &= \frac{1}{12}x^4 - \frac{1}{3}x \end{aligned}$$

If we now set $u(x,t) = v(x,t) + U(x)$, then

$$\begin{aligned} v_{xx} &= v_{tt} \\ v(x,0) &= \frac{1}{3}x - \frac{1}{12}x^4 \\ v_t(x,0) &= 0 \\ v(0,t) &= 0 \\ v_x(1,t) &= 0 \end{aligned}$$

We now use the method of separation of variables to determine $v(x,t)$.

If $X(x)T(t)$ is a solution of the homogeneous equation which satisfies the boundary conditions, then

$$\begin{aligned} X'' + \omega^2 X = 0 \ &; \ X(0) = X'(1) = 0 \\ X &= A\cos(\omega x) + B\sin(\omega x) \\ X(0) &= A = 0 \\ X' &= \omega B\cos(\omega x) \\ X'(1) &= \omega B\cos(\omega) = 0 \end{aligned}$$

and we have nontrivial solution when $\omega = \omega_n = (n + \frac{1}{2})\pi$, $n = 0, 1, 2, \ldots$, and the corresponding eigenfunctions are

$$X_n(x) = \sin(\omega_n x) \ .$$

The associated amplitude function $T_n(t)$ satisfies the equation

$$T_n'' + \omega_n^2 T_n = 0$$

so that

$$T_n(t) = a_n \cos(\omega_n t) + b_n \sin(\omega_n t) \ .$$

We look for a solution $v(x,t)$ in the form

$$v(x,t) = \sum_{n=0}^{\infty} \sin(\omega_n x) \left(a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right) \ .$$

From the initial conditions,

$$\begin{aligned}
\frac{1}{3}x - \frac{1}{12}x^4 &= v(x,0) \\
&= \sum_{n=0}^{\infty} a_n \sin(\omega_n x) \\
a_n \int_0^1 \sin^2(\omega_n x)\, dx &= \int_0^1 \left(\frac{1}{3}x - \frac{1}{12}x^4\right) \sin(\omega_n x)\, dx \\
\frac{1}{2} a_n &= -\frac{1}{\omega_n}\left(\frac{1}{3}x - \frac{1}{12}x^4\right)\cos(\omega_n x)\Big|_0^1 \\
&\quad + \frac{1}{3\omega_n}\int_0^1 (1 - x^3)\cos(\omega_n x)\, dx \\
&= 0 + \frac{1}{3\omega_n^2}(1 - x^3)\sin(\omega_n x)\Big|_0^1 \\
&\quad + \frac{1}{\omega_n^2}\int_0^1 x^2 \sin(\omega_n x)\, dx \\
&= 0 - \frac{1}{\omega_n^3}x^2\cos(\omega_n x)\Big|_0^1 \\
&\quad + \frac{2}{\omega_n^3}\int_0^1 x\cos(\omega_n x)\, dx \\
&= \frac{2}{\omega_n^4}x\sin(\omega_n x)\Big|_0^1 \\
&\quad - \frac{2}{\omega_n^4}\int_0^1 \sin(\omega_n x)\, dx \\
&= (-1)^n \frac{2}{\omega_n^4} - \frac{2}{\omega_n^5} \\
a_n &= \frac{4}{\omega_n^5}\left((-1)^n \omega_n - 1\right)
\end{aligned}$$

and

$$\begin{aligned} 0 &= v_t(x,0) \\ &= \sum_{n=0}^{\infty} b_n \omega_n \sin(\omega_n x) \\ b_n &= 0 \end{aligned}$$

Therefore

$$v(x,t) = \sum_{n=0}^{\infty} \frac{4}{\omega_n^5} \left((-1)^n \omega_n - 1\right) \sin(\omega_n x) \cos(\omega_n t)$$

and $u(x,t) = v(x,t) + \frac{1}{12}x^4 - \frac{1}{3}x$. ■

2. Find the solution of Laplace's equation ($u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$) in the wedge $0 \leq r < 1, -\frac{\pi}{3} < \theta < \frac{\pi}{3}$
with the boundary conditions $u(r, \frac{\pi}{3}) = u(r, -\frac{\pi}{3}) = 0,\ u(1,\theta) = \frac{\pi}{3} - |\theta|$.

Ans: We look first for separated solutions of the form $R(r)F(\theta)$. Since we have homogeneous boundary conditions for $\theta = \pm\frac{1}{3}\pi$, we choose the functions F as eigenfunctions, and the functions R as amplitudes.

Substituting into the equation, we have

$$\begin{aligned} F\left(R'' + \frac{1}{r}R'\right) + \frac{1}{r^2}RF'' = 0 \\ \frac{F''}{F} = -\frac{r^2}{R}\left(R'' + \frac{1}{r}R\right) = -\omega^2 \end{aligned}$$

so that the eigenfunctions satisfy

$$F'' + \omega^2 F = 0\ ;\ F(\pi/3) = F(-\pi/3) = 0$$

Writing the general solution as

$$F = A\sin(\omega(\theta + \frac{\pi}{3})) + B\cos(\omega(\theta + \frac{\pi}{3}))$$

we have

$$\begin{aligned} 0 &= F\left(-\frac{\pi}{3}\right) = B \\ 0 &= F(\left(\frac{\pi}{3}\right) = A\sin\left(\frac{2\pi\omega}{3}\right) \end{aligned}$$

so that the eigenvalues satisfy

$$\frac{2\pi\omega_n}{3} = n\pi\ ; \omega_n = \frac{3n}{2}\ ; n = 1, 2, \dots\ .$$

The amplitude functions satisfy the Euler equation

$$r^2 R'' + rR' - \omega_n^2 R = 0$$

for which the indicial equation is

$$l(l-1)+l-\omega_n^2=0\ ;\ l^2=\omega_m^2$$

so that the general solution is

$$R(r)=Ar^{\omega_n}+Br^{-\omega_n}$$

However, the solution vanishes when $r=0$, so that $B=0$.

We therefore look for a solution in the form

$$u(r,\theta)=\sum_{n=1}^{\infty}a_n r^{3n/2}\sin\left(\frac{3n}{2}\left(\theta+\frac{\pi}{3}\right)\right)$$

where the coefficients are determined by the boundary condition

$$\sum_{n=1}^{\infty}a_n\sin\left(\frac{3n}{2}\left(\theta+\frac{\pi}{3}\right)\right)=\frac{\pi}{3}-|\theta|$$

$$a_n\int_{-\pi/3}^{\pi/3}\sin^2\left(\frac{3n}{2}\left(\theta+\frac{\pi}{3}\right)\right)\,d\theta$$

$$=\int_{-\pi/3}^{0}(\frac{\pi}{3}+\theta)\sin(\left(\frac{3n}{2}\left(\theta+\frac{\pi}{3}\right)\right)\,d\theta$$

$$+\int_{0}^{\pi/3}(\frac{\pi}{3}-\theta)\sin(\left(\frac{3n}{2}\left(\theta+\frac{\pi}{3}\right)\right)\,d\theta$$

$$=-\frac{2}{3n}(\theta+\pi/3)\cos(\left(\frac{3n}{2}\left(\theta+\frac{\pi}{3}\right)\right)\Big|_{-\pi/3}^{0}+\frac{2}{3n}\int_{-\pi/3}^{0}\cos(\left(\frac{3n}{2}\left(\theta+\frac{\pi}{3}\right)\right)\,d\theta$$

$$-\frac{2}{3n}(-\theta+\pi/3)\cos(\left(\frac{3n}{2}\left(\theta+\frac{\pi}{3}\right)\right)\Big|_{0}^{\pi/3}-\frac{2}{3n}\int_{0}^{\pi/3}\cos(\left(\frac{3n}{2}\left(\theta+\frac{\pi}{3}\right)\right)\,d\theta$$

$$=-\frac{2\pi}{9n}\cos\left(\frac{n\pi}{2}\right)+\frac{2\pi}{9n}\cos\left(\frac{n\pi}{2}\right)$$

$$+\frac{4}{9n^2}\sin\left(\frac{3n}{2}\left(\theta+\frac{\pi}{3}\right)\right)\Big|_{-\pi/3}^{0}-\frac{4}{9n^2}\sin\left(\frac{3n}{2}\left(\theta+\frac{\pi}{3}\right)\right)\Big|_{0}^{\pi/3}$$

$$=\frac{8}{9n^2}\sin\left(\frac{n\pi}{2}\right)$$

Therefore, $a_n=0$ if n is even, while when $n=2m+1$,

$$a_{2m+1}=\frac{8}{3\pi}\frac{1}{(2m+1)^2}(-1)^m$$

$$u(r,\theta)=\frac{8}{3\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}r^{3m+3/2}\sin((3m+3/2)\theta+(m+1/2)\pi)$$

$$=\frac{8}{3\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}r^{3m3/2}\cos((3m+3/2)\theta)$$

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3. Find the solution of Laplace's equation ($u_{xx} + u_{yy} = 0$) for $0 < x < 1,\ 0 < y < 1$

with the boundary conditions

$u(1, x) = x - \frac{1}{2}x^2,\ u(0, x) = 0,\ u(y, 0) = 0,\ u_x(y, 1) = 0.$

Ans: Note : for some obscure reason, the unknown function has been written as $u(y, x)$.

We look for separated solutions $X(x)Y(y)$ where $X(0) = X'(1) = 0$.

The eigenfunctions satisfy

$$X'' + \omega^2 X = 0\ ; X(0) = X'(1) = 0$$
$$X = A\cos(\omega x) + B\sin(\omega x)$$
$$0 = X(0) = AX'(x) = \omega B\cos(\omega x)$$
$$0 = X'(1) = \omega B\cos(\omega)$$

so that the eigenvalues are given by $\omega = \omega_n = (n + \frac{1}{2})\pi,\ n = 0, 1, 2, \ldots$, and the eigenfunctions by $X_n(x) = \sin(\omega_n x)$.

The corresponding amplitude functions Y_n satisfy

$$Y_n'' - \omega_n^2 Y_n = 0$$
$$Y_n = a_n\cosh(\omega_n y) + b_n\sinh(\omega_n y)$$

We look for a solution in the form

$$u(y, x) = \sum_{n=0}^{\infty}(a_n\cosh(\omega_n y) + b_n\sinh(\omega_n y))\sin(\omega_n x)$$

where

$$0 = u(0, x) = \sum_{n=0}^{\infty} a_n\sin(\omega_n x)$$
$$a_n = 0\ \forall\, n$$
$$x - \frac{1}{2}x^2 = u(1, x) = \sum_{n=0}^{\infty} b_n\sinh(\omega_n)\sin(\omega_n x)$$

$$b_n\sinh(\omega_n)\int_0^1 \sin^2(\omega_n x)\,dx = \int_0^1\left(x - \frac{1}{2}x^2\right)\sin(\omega_n x)\,dx$$
$$\frac{\sinh(\omega_n)}{2}b_n = -\frac{1}{\omega_n}\left(x - \frac{1}{2}x^2\right)\cos(\omega_n x)\Big|_0^1 + \frac{1}{\omega_n}\int_0^1(1 - x)\cos(\omega_n x)\,dx$$
$$= 0 + \frac{1}{\omega_n^2}(1 - x)\sin(\omega_n x)\Big|_0^1 + \frac{1}{\omega_n^2}\int_0^1\sin(\omega_n x)\,dx$$
$$= 0 - \frac{1}{\omega_n^3}\cos(\omega_n x)\Big|_0^1 = \frac{1}{\omega_n^3}$$
$$u(y, x) = \sum_{n=0}^{\infty}\frac{2}{\omega_n^3}\frac{\sinh(\omega_n y)}{\sinh(\omega_n)}\sin(\omega_n x)\ .$$

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