MATH3403

Tutorial Sheet 7

Solutions

1. Show that
$$\frac{d}{dx}\left(x^{\nu}J_{\nu}(x)\right) = x_{\nu}J_{\nu-1}(x)\ ,$$
and that
$$\frac{d}{dx}\left(x^{-\nu}J_{\nu}(x)\right) = -x_{-\nu}J_{\nu+1}(x)\ .$$

Hence express $J_\nu'(x)$ in terms of $J_\nu(x)$ and $J_{\nu-1}(x)$, and also interms of $J_\nu(x)$ and $J_{\nu+1}(x)$.

Combine these results to obtain a three term recurrence relation connecting $J_{\nu-1}(x)$, $J_\nu(x)$ and $J_{\nu+1}(x)$.

Ans

$$\begin{aligned}
\frac{d}{dx}\left(x^{\nu}J_{\nu}(x)\right) &= \frac{d}{dx}2^{\nu}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+1+\nu)}\left(\frac{x}{2}\right)^{2n+2\nu} \\
&= 2^{\nu}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+1+\nu)}\frac{2n+2\nu}{2}\left(\frac{x}{2}\right)^{2n+2\nu-1} \\
&= 2^{\nu}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\nu)}\left(\frac{x}{2}\right)^{2n+2\nu-1} \\
&= x^{\nu}J_{\nu-1}(x)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\left(x^{-\nu}J_{\nu}(x)\right) &= \frac{d}{dx}2^{-\nu}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+1+\nu)}\left(\frac{x}{2}\right)^{2n} \\
&= 2^{-\nu}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+1+\nu)}\frac{2n}{2}\left(\frac{x}{2}\right)^{2n-1} \\
&= 2^{-\nu}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)!\Gamma(n+1+\nu)}\left(\frac{x}{2}\right)^{2n-1} \\
&= 2^{-\nu}\sum_{p=0}^{\infty}\frac{(-1)^{p+1}}{p!\Gamma(p+2+\nu)}\left(\frac{x}{2}\right)^{2p+1} \\
&= -x^{-\nu}J_{\nu+1}(x)
\end{aligned}$$

$$\begin{aligned}
x^{\nu}J_\nu'(x) + \nu x^{\nu-1}J_\nu(x) &= x^{\nu}J_{\nu-1}(x) \\
J_\nu'(x) &= J_{\nu-1}(x) - \frac{\nu}{x}J_\nu(x) \\
x^{-\nu}J_\nu'(x) - \nu x^{-\nu-1}J_\nu(x) &= x^{-\nu}J_{\nu+1}(x) \\
J_\nu'(x) &= -J_{\nu+1}(x) + \frac{\nu}{x}J_\nu(x) \\
0 &= J_{\nu-1}(x) - 2\frac{\nu}{x}J_\nu(x) + J_{\nu+1}(x)
\end{aligned}$$

In particular

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

(see question 3)

2. Show that $y = J_n(\lambda x)$ satisfies the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2 x^2 - n^2)y = 0$$

Ans. If $t = \lambda x$, $y = J_n(t)$ satisfies

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - n^2)y = 0$$

Now

$$\frac{d}{dt} = \frac{dx}{dt}\frac{d}{dx} = \frac{1}{\lambda}\frac{d}{dx}$$
$$t\frac{d}{dt} = (\lambda x)\frac{1}{\lambda}\frac{d}{dx} = x\frac{d}{dx}$$
$$\frac{d^2}{dt^2} = \frac{1}{\lambda^2}\frac{d^2}{dx^2}$$
$$t^2\frac{d^2}{dt^2} = x^2\frac{d^2}{dx^2}$$

Substituting these results into the differential equation gives

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2 x^2 - n^2)y = 0$$

as required.

Hence show that, for $\lambda \neq \mu$,

$$(\lambda^2 - \mu^2)\int_0^a xJ_n(\lambda x)J_n(\mu x)\,dx = a\left[\mu J_n(\lambda a)J_n'(\mu a) - \lambda J_n(\mu a)J_n'(\lambda a)\right]$$

Ans. Dividing the result above through by x and rearranging we have

$$\left(\lambda^2 x - \frac{n^2}{x}\right)J_n(\lambda x) = -\frac{d}{dx}\left(x\frac{dJ_n(\lambda x)}{dx}\right)$$
$$\left(\mu^2 x - \frac{n^2}{x}\right)J_n(\mu x) = -\frac{d}{dx}\left(x\frac{dJ_n(\mu x)}{dx}\right)$$

Multiplying the first equation by $J_n(\mu x)$ and the second by $J_n(\lambda x)$ and subtracting we get

$$(\lambda^2 - \mu^2)xJ_n(\lambda_x)J_n(\mu x) = -\frac{d}{dx}\left(x\frac{dJ_n(\lambda x)}{dx}\right)J_n(\mu x) + \frac{d}{dx}\left(x\frac{dJ_n(\mu x)}{dx}\right)J_n(\lambda x)$$

Finally, integrating this from 0 to a gives

$$\begin{aligned}(\lambda^2 - \mu^2)\int_0^a xJ_n(\lambda x)J_n(\mu x)\,dx &= \int_0^a\left(-\frac{d}{dx}\left(x\frac{dJ_n(\lambda x)}{dx}\right)J_n(\mu x) + \frac{d}{dx}\left(x\frac{dJ_n(\mu x)}{dx}\right)J_n(\lambda x)\right)dx \\ &= \left[-x\frac{dJ_n(\lambda x)}{dx}J_n(\mu x) + x\frac{dJ_n(\mu x)}{dx}J_n(\lambda x)\right]_0^a \\ &= a\left[\mu J_n'(\mu a)J_n(\lambda a) - \lambda J_n'(\lambda a)J_n(\mu a)\right]\end{aligned}$$

Use L'Hôpital's rule to show that

$$\int_0^a x\,(J_n(\mu x))^2\;dx = \frac{1}{2}a^2\,(J_n'(\mu a))^2 + \frac{1}{2}\left(a^2 - \frac{n^2}{\mu^2}\right)(J_n(\mu a))^2$$

Ans. Dividing both sides of the above result by $\lambda^2 - \mu^2$ and then differentiating the top and bottom of the resulting fraction with respect to λ gives

$$\begin{aligned}\int_0^a x\,(J_n(\mu x))^2\;dx =& \lim_{\lambda\to\mu}\frac{1}{2\lambda}\frac{d}{d\lambda}a\,(\mu J_n'(\mu a)J_n(\lambda a) - \lambda J_n'(\lambda a)J_n(\mu a))\\ =&\frac{a}{2\mu}\left[a\mu J_n'(\mu a)J_n'(\lambda a) - J_n'(\lambda a)J_n(\mu a) - \lambda a J_n''(\lambda a)J_n(\mu a)\right]_{\lambda=\mu}\end{aligned}$$

Now

$$\lambda a J_n''(\lambda a) + J_n'(\lambda a) = \left(\frac{n^2}{\lambda a} - \lambda a\right)J_n(\lambda a)$$

so that

$$\begin{aligned}\int_0^a x\,(J_n(\mu x))^2\;dx &= \frac{a^2}{2}\,(J'(\mu a))^2 + \frac{a}{2\mu}\left(\mu a - \frac{n^2}{\mu a}\right)(J_n(\mu a))^2\\ &= \frac{1}{2}a^2\,(J_n'(\mu a))^2 + \frac{1}{2}\left(a^2 - \frac{n^2}{\mu^2}\right)(J_n(\mu a))^2\end{aligned}$$

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3. Determine the solution of the cylindrical wave equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}$$
$$t \ge 0\;;\;\; 0 \le r < 1$$

with the initial conditions

$$u(r,0) = \epsilon(1 - r^2)\;;u_t(r,0) = 0$$

and the boundary condition

$$u(1,t) = 0$$

Ans. Consider the more general problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}$$
$$t \ge 0\;;\;\; 0 \le r < 1$$

with the boundary condition

$$u(1,t) = 0$$

(and the implicit requirement that the solution be finite for $0 \le r < 1$.)

Writing $u = R(r)T(t)$, we have

$$R''T + \frac{1}{r}R'T = RT''$$
$$\frac{1}{R}\left(R'' + \frac{1}{r}R'\right) = \frac{T''}{T} = -\omega^2$$

which gives the equation

$$R'' + \frac{1}{r}R' + \omega^2 R = 0\ ,\ \ R(1) = 0$$

for the eigenfunctions.

The general solution of this equation is $AJ_0(\omega r) + BY_0(\omega r)$.

Since we require that R be finite when $r = 0$, $B = 0$.

The requirement $R(1) = 0$ gives the equation $J_0(\omega) = 0$.

Therefore the eigenvalues are given by $\omega_n = j_{0,n}$, and the corresponding eigenfunctions are $J_0(\omega_n r)$.

For each ω_n, $T'' + \omega_n^2 T = 0$, which gives

$$T_n(t) = a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\ .$$

The separated solutions which satisfy $u(1, t) = 0$ are

$$J_0(\omega_n r)(a_n \cos(\omega_n t) + b_n \sin(\omega_n t))\ ,$$

and the general solution satisfying this boundary condition can be written as

$$u(r, t) = \sum_{n=1}^{\infty} J_0(\omega_n r)(a_n \cos(\omega_n t) + b_n \sin(\omega_n t))\ .$$

Returning to the original problem, we can choose the unknown coefficients a_n, b_n to satisfy the initial conditions:

$$u(r, 0) = \sum_{n=1}^{\infty} a_n J_0(\omega_n r) = \epsilon(1 - r^2)$$

$$a_n \int_0^1 r J_0^2(\omega_n r)\,dr = \epsilon \int_0^1 (1 - r^2) r J_0(\omega_n r)\,dr$$
$$\frac{1}{2\epsilon} J_1^2(\omega_n) a_n = (1 - r^2)\frac{1}{\omega_n} r J_1(\omega_n r)\Big|_0^1$$
$$\frac{2}{\omega_n} \int_0^1 r^2 J_1(\omega_n r)\,dr$$
$$= \frac{2}{\omega_n^2} J_2(\omega_n r)\Big|_0^1$$
$$= \frac{2}{\omega_n^2} J_2(\omega_n) = \frac{4}{\omega_n^3} J_1(\omega_n)$$
$$a_n = \frac{8\epsilon}{\omega_n^3 J_1(\omega_n)}$$

$$u_t(r,t) = \sum_{n=1}^{\infty} J_0(\omega_n r)(-a_n\omega_n \sin(\omega_n t) + b_n\omega_n \cos(\omega_n t))$$

$$u_t(r,0) = \sum_{n=1}^{\infty} b_n\omega_n J_0(\omega_n r) = 0$$

$$b_n = 0$$

Therefore the required solution is

$$u(r,t) = 8\epsilon \sum_{n=1}^{\infty} \frac{1}{\omega_n^3 J_1(\omega_n)} J_0(\omega_n r)\cos(\omega_n t)\ .$$

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