MATH 3403

TUTORIAL SHEET 6

SOLUTIONS

1. Find a weight factor which will convert the following operators to self adjount form.

(a) Tchebychev's operator

$$\mathcal{T}(y) = \left(1 - x^2\right) \frac{d^2y}{dx^2} - x\,\frac{dy}{dx}$$

(b) Laguerre's operator

$$\mathcal{L}(y) = x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx}$$

(c) Hermite's operator

$$\mathcal{H}(y) = \frac{d^2y}{dx^2} - 2x\frac{dy}{dx}$$

(d) The cylindrical Laplace operator

$$\mathcal{C}(y) = \frac{d^2y}{dr^2} + \frac{1}{r}\,\frac{dy}{dr}$$

Ans

(a)
$$\begin{aligned}\frac{w'}{w} &= \frac{p_1 - p_2'}{p_2} \\ &= \frac{-x + 2x}{1 - x^2} = \frac{x}{1-x^2} \\ \log(w) &= -\frac{1}{2}\log(1 - x^2) \\ w &= \frac{1}{\sqrt{1-x^2}}\end{aligned}$$

(b)
$$\begin{aligned}\frac{w'}{w} &= \frac{p_1 - p_2'}{p_2} \\ &= \frac{1 - x - 1}{x} = -1 \\ w' &= -w \;;\; w = e^{-x}\end{aligned}$$

(c)
$$\begin{aligned}\frac{w'}{w} &= \frac{p_1 - p_2'}{p_2} \\ &= -2x \\ \log(w) &= -x^2 \;;\; w = e^{-x^2}\end{aligned}$$

(d)
$$\begin{aligned}\frac{w'}{w} &= \frac{p_1 - p_2'}{p_2} \\ &= \frac{1}{r} \\ \log(w) &= \log r \;; w = r\end{aligned}$$

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2. The Tchebychev operator, (1(a)), has eigenfunctions $T_n(x)$ on the interval $[-1,1]$ which are polynomials of degree n in x;

$$T_n(x) = \sum_{r=0}^{n} a_r x^r \ ; \ n = 0, 1, 2, \ldots$$

scaled so that $T_n(1) = 1$.

Determine the first four eigenfunctions and the corresponding eigenvalues.

Ans The eigenfunction equation is

$$(1 - x^2)y'' - xy' + \lambda y = 0$$

Setting $y = \sum_{r=0}^{\infty} a_r x^r$, we obtain

$$\sum_{r=0}^{\infty} r(r-1)a_r x^{r-2} - \sum_{r=0}^{\infty} r(r-1)a_r x^r - \sum_{r=0}^{\infty} r a_r x^r + \sum_{r=0}^{\infty} \lambda a_r x^r = 0$$

$$\sum_{r=0}^{\infty}(r+2)(r+1)a_{r+2}x^r = \sum_{r=0}^{\infty}(r^2 - \lambda)a_r x^r$$

$$a_{r+2} = \frac{r^2 - \lambda}{(r+1)(r+2)} a_r$$

The expansion terminates whenever $\lambda = r^2$ for some integer r. Therefore we get the polynomial solutions when $\lambda = 0, 1, 4, 9, \ldots$.

When $\lambda = 0$, $T_0(x) = 1$.

When $\lambda = 1$, $T_1(x) = x$.

When $\lambda = 4$,

$$a_2 = \frac{-4}{1 \times 2} a_0 = -2a_0$$

$$T_2(x) = a_0(1 - 2x^2) \ ; \ T_2(1) = -a_0$$

$$T_2(x) = 2x^2 - 1$$

When $\lambda = 9$,

$$a_3 = \frac{1-9}{2 \times 3} a_1 = -\frac{4}{3} a_1$$

$$T_3(x) = a_1(x - \frac{4}{3}x^3) \ ; T_3(1) = -\frac{1}{3} a_1$$

$$T_3(x) = 4x^3 - 3x$$

Do these numbers remind you of anything? ■

3. Solve the following Euler equations for $x > 0$.

(a) $$x^2 y'' + 2xy' - 2y = 0$$

(b) $$x^2 y'' - xy' - 3y = 0$$

(c) $$2x^2 y'' + xy' - y = 0$$

(d) $$x^2 y'' + xy + y = 0$$

Ans

(a) The indicial equation is

$$\begin{aligned} r(r-1)+2r-2&=0 \\ r^2+r-2&=0 \\ (r+2)(r-1)&=0 \\ r&=1,-2 \end{aligned}$$

Therefore the general solution is

$$y = Ax + Bx^{-2}$$

(b) The indicial equation is

$$\begin{aligned} r(r-1)-r-3&=0 \\ r^2-2r-3&=0 \\ (r-3)(r+1)&=0 \\ r&=3,-1 \end{aligned}$$

Therefore the general solution is

$$y = Ax^3 + Bx^{-1}$$

(c) The indicial equation is

$$\begin{aligned} 2r(r-1)+r-1&=0 \\ 2r^2-r-1&=0 \\ (2r+1)(r-1)&=0 \\ r&=1,-\frac{1}{2} \end{aligned}$$

Therefore the general solution is

$$y = Ax + \frac{B}{\sqrt{x}}$$

(d) The indicial equation is

$$\begin{aligned} r(r-1)+r+1&=0 \\ r^2+1&=0 \\ r&=\pm i \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= ax^i + bx^{-i} \\ &= ae^{i\log x} + be^{-i\log x} \\ &= (a+b)\cos(\log x) + i(a-b)\sin(\log x) \\ &= A\cos(\log x) + B\sin(\log x) \end{aligned}$$

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