MATH 3403

Tutorial Sheet 5

Solutions

1. Determine the eigenvalues and eigenfunctions for the following problems.

(a) $$X'' + \lambda X = 0 \; ; X'(0) = X'(1) = 0$$

Ans For $\lambda > 0$, set $\lambda = \omega^2$. This gives the simple harmonic equation

$$X'' + \omega^2 X = 0$$

for which the general solution is

$$\begin{aligned} X(x) &= A\cos(\omega x) + B\sin(\omega x) \\ X'(x) &= -\omega A\sin(\omega x) + \omega B\cos(\omega x) \end{aligned}$$

The boundary condition $X'(0) = 0$ implies $B = 0$, and the other boundary condition gives $A\sin(\omega) = 0$.

For non-trivial solutions we require $A \neq 0$, so that we have eigenvalues and eigen functions when $\sin(\omega) = 0$, which gives

$$\omega = n\pi \; ; \; \lambda = n^2\pi^2 \; ; n = 1, 2, \ldots$$

The corresponding eigenfunctions are

$$X_n(x) = \cos(n\pi x) \; .$$

For $\lambda < 0$, set $\lambda = -p^2$. This gives the equation

$$X'' - p^2 X = 0$$

for which the general solution is

$$\begin{aligned} X(x) &= A\cosh(px) + B\sinh(px) \\ X'(x) &= pA\sinh(px) + pB\cosh(px) \end{aligned}$$

The boundary condition $X'(0) = 0$ implies $B = 0$, and the other boundary condition gives $A\sinh(p) = 0$.

Since $\sinh(p) \neq 0$ for $p > 0$, $A = 0$, and there are no negative eigenvalues.

For $\lambda = 0$, the equation reduces to

$$X'' = 0$$

for which the general solution is

$$\begin{aligned} X(x) &= A + Bx \\ X'(x) &= B \end{aligned}$$

The boundary conditions are satisfied by $B = 0$, leaving A arbitrary, so that $\lambda = 0$ is an eigenvalue, and the corresponding eigenfunction is

$$X_0(x) = 1 \; .$$

(c.f. Question 2) ■

(b) $$X'' + \lambda X = 0 \;; X(0) = 0 \;,\; X'(1) = X(1)$$

Ans For $\lambda > 0$, set $\lambda = \omega^2$. This gives the simple harmonic equation

$$X'' + \omega^2 X = 0$$

for which the general solution is

$$\begin{aligned} X(x) &= A\cos(\omega x) + B\sin(\omega x) \\ X'(x) &= -\omega A\sin(\omega x) + \omega B\cos(\omega x) \end{aligned}$$

The boundary condition $X(0) = 0$ implies $A = 0$, and the other boundary condition gives

$$B\sin(\omega) = B\omega\cos(\omega) \;.$$

For non-trivial solutions we require $B \neq 0$, so that we have eigenvalues and eigen functions when $\tan(\omega) = \omega$, which gives

$$\omega_n \sim \left(n + \frac{1}{2}\right)\pi \; - \frac{1}{\left(n + \frac{1}{2}\right)\pi}; n = 1, 2, \ldots$$

The corresponding eigenfunctions are

$$X_n(x) = \sin(\omega_n x) \;.$$

For $\lambda < 0$, set $\lambda = -p^2$. This gives the equation

$$X'' - p^2 X = 0$$

for which the general solution is

$$\begin{aligned} X(x) &= A\cosh(px) + B\sinh(px) \\ X'(x) &= pA\sinh(px) + pB\cosh(px) \end{aligned}$$

The boundary condition $X(0) = 0$ implies $A = 0$, and the other boundary condition gives $B\sinh(p) = pB\cosh(p)$.

Since $\tanh(p) < p$ for $p > 0$, $B = 0$, and there are no negative eigenvalues.

For $\lambda = 0$, the equation reduces to

$$X'' = 0$$

for which the general solution is

$$\begin{aligned} X(x) &= A + Bx \\ X'(x) &= B \end{aligned}$$

The boundary condition $X(0) = 0$, implies $A = 0$, while the other condition gives $B = B$, which is true, so that $\lambda = 0$ is an eigenvalue, and the corresponding eigenfunction is

$$X_0(x) = x \;.$$

■

(c) $$x^2X'' + xX' + \lambda X = 0 \; ; X(1) = X(2) = 0$$

Ans This equation is an Euler equation. Setting $X = x^r$, we obtain the indicial equation

$$r(r-1) + r + \lambda = 0 \; ; r^2 + \lambda = 0$$

When $\lambda < 0$, set $\lambda = -p^2$, which gives $r = \pm p$.
The general solution of the equation is

$$X(x) = Ax^p + Bx^{-p}$$

The boundary conditions give

$$\begin{aligned} A + B &= 0 \\ 2^pA + 2^{-p}B &= 0 \end{aligned}$$

for which the only solution is $A = 0$, $B = 0$. Therefore there are no negative eigenvalues.
When $\lambda = 0$, the general solution of the equation is

$$X(x) = A + B\log x$$

The boundary conditions give

$$A = 0 \ , \ B\log 2 = 0$$

so that $\lambda = 0$ is not an eigenvalue.
Finally, for $\lambda > 0$, set $\lambda = \omega^2$. The general solution is

$$X(x) = A\sin(\omega \log x) + B\cos(\omega \log x)$$

The condition $X(1) = 0$ implies $B = 0$, while the condition $X(2) = 0$ gives

$$A\sin(\omega \log 2) = 0$$

so that we get non-trivial solutions when

$$\omega_n = \frac{n\pi}{\log 2} \; ; \; n = 1, 2, \ldots$$

and the corresponding eigenfunctions are

$$X_n = \ \sin\left(\frac{n\pi}{\log 2}\log x\right) \ .$$

■

2. Verify that the functions 1, $\{\cos(nx)\}$ $n = 1, 2, \ldots,$ are orthogonal on the interval $[0, \pi]$.

Ans The functions X_m, X_n are orthogonal if

$$\int_0^\pi X_m X_n \, dx = 0 \;; m \neq n$$

For $X_m = 1$ and $X_n = \cos(nx)$, we have

$$\begin{aligned}\int_0^\pi X_m X_n \, dx &= \int_0^\pi \cos(nx) \, dx \\ &= \frac{1}{n} \sin(nx) \Big|_0^\pi = 0\end{aligned}$$

and for $X_m = \cos(mx)$ and $X_n = \cos(nx)$, we have

$$\begin{aligned}\int_0^\pi X_m X_n \, dx &= \int_0^\pi \cos(mx)\cos(nx) \, dx \\ &= \frac{1}{2} \int_0^\pi \left(\cos((m+n)x) + \cos((m-n)x)\right) \, dx \\ &= \frac{1}{2} \left(\frac{1}{m+n} \sin((m+n)x) + \frac{1}{m-n} \sin((m-n)x) \right) \Big|_0^\pi = 0\end{aligned}$$

■

Expand the function $f(x) = \sinh x$, $0 \leq x \leq \pi$, in the series form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \; .$$

Ans Since the functions are orthogonal on the interval, we have

$$\begin{aligned}a_0 \int_0^\pi 1^2 \, dx &= \int_0^\pi f(x).1 \, dx \\ \pi a_0 &= \cosh x|_0^\pi = \cosh(\pi) - 1 \\ a_0 &= \frac{\cosh(\pi) - 1}{\pi}\end{aligned}$$

$$\begin{aligned}a_n \int_0^\pi \cos^2(nx) \, dx &= \int_0^\pi \sinh(x)\cos(nx) \, dx \\ \int_0^\pi \cos^2(nx) \, dx &= \frac{1}{2} \int_0^\pi \left(1 + \cos(2nx)\right) \, dx \\ &= \frac{1}{2} \left(x + \frac{1}{2n} \sin(2nx) \right) \Big|_0^\pi \\ &= \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned} I = \int_0^\pi \sinh(x)\cos(nx)\,dx =&\ \cosh(x)\cos(nx)|_0^\pi + n\int_0^\pi \cosh(x)\sin(nx)\,dx \\ =&\ ((-1)^n\cosh(\pi) - 1) + n\sinh(x)\sin(nx)|_0^\pi \\ &- n^2\int_0^\pi \sinh(x)\cos(nx)\,dx \\ (n^2+1)I =&\ ((-1)^n\cosh(\pi) - 1) \\ I =&\ \frac{1}{n^2+1}((-1)^n\cosh(\pi) - 1) \\ a_n =&\ \frac{2}{\pi(n^2+1)}((-1)^n\cosh(\pi) - 1) \end{aligned}$$

■

3. Find the solution of the finite wave equation

$$u_{xx} = u_{tt}\ ;\ 0 \le x \le 1\ ;\ t \ge 0$$

which satisfies the initial conditions

$$u(x,0) = 2x - x^2\ ;\ u_t(x,0) = 0$$

and the boundary conditions

$$u(0,t) = 0\ ;\ u_x(1,t) = 0\ .$$

Ans If we look for separated solutions of the form $u = X(x)T(t)$, then the harmonics $X(x)$ are defined by

$$X'' + \lambda X = 0\ ;\ X(0) = 0\ ,\ X'(1) = 0\ .$$

For $\lambda > 0$, we set $\lambda = \omega^2$, obtaining the simple harmonic equation

$$X'' + \omega^2 X = 0$$

for which the general solution is

$$\begin{aligned} X =&\ A\cos(\omega x) + B\sin(\omega x) \\ X' =&\ -\omega A\sin(\omega x) + \omega B\cos(\omega x) \end{aligned}$$

The boundary condition $X(0) = 0$ implies $A = 0$, while $X'(1) = 0$ gives

$$B\cos(\omega) = 0$$

and we have nontrivial solutions when

$$\omega_n = \left(n + \frac{1}{2}\right)\pi\ ;\ n = 0, 1, \ldots$$

and the eigenfunctions are $\sin(\omega_n x)$.

Since $\sin(\frac{1}{2}\pi x)$ does not cross the axis in $(0,1)$, it is the fundamental, and there are no non-positive eigenvalues.

The amplitude functions $T(t)$ satisfy

$$T'' + w_n^2 T = 0$$

giving

$$T_n(t) = a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$$

and the solution expansion

$$u(x,t) = \sum_{n=0}^{\infty} \sin(\omega_n x) \left(a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right) \ .$$

From the initial condition $u_t(x,0) = 0$, we have $b_n = 0$ for all n, while the other initial condition gives

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n \sin(\omega_n x) &= 2x - x^2 \\
a_n \int_0^1 \sin^2(\omega_n x)\, dx &= \int_0^1 (2x - x^2) \sin(\omega_n x)\, dx \\
\frac{1}{2} a_n &= -(2x - x^2) \frac{1}{\omega_n} \cos(\omega_n x) \Big|_0^1 + \frac{1}{\omega_n} \int_0^1 (2 - 2x) \cos(\omega_n x)\, dx \\
&= 0 + \frac{1}{\omega_n^2} (2 - 2x) \sin(\omega_n x) \Big|_0^1 + \frac{2}{\omega_n^2} \int_0^1 \sin(\omega_n x)\, dx \\
&= 0 - \frac{2}{\omega_n^3} \cos(\omega_n x) \Big|_0^1 \\
&= \frac{2}{\omega_n^3}
\end{aligned}$$

so that

$$u(x,t) = \sum_{n=0}^{\infty} \frac{4}{\omega_n^3} \sin(\omega_n x) \cos(\omega_n t) \ .$$

■