

MATH 3403
TUTORIAL SHEET 4
SOLUTIONS

1. Find the solution to

$$u_{xx} = 4u_{tt}, \quad 0 < t < \infty, \quad 0 < x < \infty$$

with initial conditions

$$\begin{aligned} u(x, 0) &= \sin x, & x > 0 \\ u_t(x, 0) &= -\cos x, & x > 0 \end{aligned}$$

and boundary condition

$$u_x(0, t) = 0, \quad t > 0.$$

Ans Since $c^{-2} = 4$, $c = \frac{1}{2}$.

In the region $x > \frac{1}{2}t$, we have d'Alembert's solution

$$\begin{aligned} u(x, t) &= \frac{1}{2} \left(\sin \left(x + \frac{1}{2}t \right) + \sin \left(x - \frac{1}{2}t \right) \right) - \int_{x-t/2}^{x+t/2} \cos s \, ds \\ &= \frac{1}{2} \left(\sin \left(x + \frac{1}{2}t \right) + \sin \left(x - \frac{1}{2}t \right) \right) - \sin \left(x + \frac{1}{2}t \right) + \sin \left(x - \frac{1}{2}t \right) \\ &= -\frac{1}{2} \sin \left(x + \frac{1}{2}t \right) + \frac{3}{2} \sin \left(x - \frac{1}{2}t \right) \end{aligned}$$

In the region $0 < x < \frac{1}{2}t$, we have

$$\begin{aligned} u(x, t) &= \phi(t - 2x) - \frac{1}{2} \sin \left(x + \frac{1}{2}t \right) \\ u_x(x, t) &= -2\phi'(t - 2x) - \frac{1}{2} \cos \left(x + \frac{1}{2}t \right) \\ u_x(0, t) &= -2\phi'(t) - \frac{1}{2} \cos \left(\frac{t}{2} \right) \\ &= 0 \\ \phi'(t) &= -\frac{1}{4} \cos \left(\frac{t}{2} \right) \\ \phi(t) &= -\frac{1}{2} \sin \left(\frac{t}{2} \right) + c \\ \text{where } c &= 0 \text{ since } u(0, 0) = 0 \\ u(x, t) &= -\frac{1}{2} \left(\sin \left(\frac{t}{2} - x \right) + \sin \left(\frac{t}{2} + x \right) \right) \end{aligned}$$

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2. Find $u(x, t)$ for $t > 0$, $-\infty < x < \infty$, if

$$\begin{cases} u_{xx} = u_{tt} & x < 0 \\ u_{xx} = 4u_{tt} & x > 0 \\ u(x, 0) = \sin x & x < 0 \\ u(x, 0) = 0 & x > 0 \\ u_t(x, 0) = 0 \end{cases}$$

Ans This is a transmission problem. There are two separate partial differential equations, one for $x < 0$ and the other for $x > 0$, and four regions of interest.

For $x < 0$, we have $c = 1$.

In the region $x < -t$, the solution is determined by the initial data $u(x, 0) = \sin x$, $u_t(x, 0) = 0$.

D'Alembert's formula gives the solution

$$u(x, t) = \frac{1}{2} \sin(x - t) + \frac{1}{2} \sin(x + t)$$

In the region $-t < x < 0$, we have the right moving wave $\frac{1}{2} \sin(x - t)$ from the previous solution together with a left moving wave $\psi(t + x)$ which is yet to be determined.

$$u(x, t) = \frac{1}{2} \sin(x - t) + \psi(t + x)$$

For $x > 0$, we have $c = \frac{1}{2}$.

In the region $x > \frac{1}{2}t$, the solution is determined by the initial data $u(x, 0) = 0$, $u_t(x, 0) = 0$. Hence $u(x, t) = 0$, and in particular there is no left moving wave.

Therefore, in the region $0 < x < \frac{1}{2}t$, the solution consists of the transmitted right moving wave $\phi(t - 2x)$.

Along the line $x = 0$ we expect that the solution will be continuous.

Therefore

$$(1) \quad \frac{1}{2} \sin(-t) + \psi(t) = \phi(t)$$

We also expect that the derivative with respect to x will be continuous.

For $x < 0$,

$$u_x(x, t) = \frac{1}{2} \cos(x - t) + \psi'(t + x)$$

while for $x > 0$,

$$u_x(x, t) = -2\phi'(t - 2x)$$

Equating these when $x = 0$ gives

$$\frac{1}{2} \cos(-t) + \psi'(t) = -2\phi'(t)$$

Integrating this we obtain

$$(2) \quad \frac{1}{2} \sin(t) + \psi(t) = -2\phi(t)$$

where we have use the fact that $u(0,0) = 0$, so that $\phi(0) = \psi(0) = 0$.

From (1) and (2) we obtain

$$\phi(t) = -\frac{1}{3}\sin(t) ; \psi(t) = \frac{1}{6}\sin(t)$$

so that, for $-t < x < 0$,

$$u(x,t) = \frac{1}{2}\sin(x-t) + \frac{1}{6}\sin(x+t)$$

and for $0 < x < \frac{1}{2}t$,

$$u(x,t) = -\frac{1}{3}\sin(t-2x) .$$

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3. Solve the moving boundary problem

$$\begin{aligned} u_{xx} - u_{tt} &= 0 ; \quad 0 < t < \infty , \quad \frac{1}{2}t < x < \infty \\ u(x,0) &= e^{-x} , \quad u_t(x,0) = e^{-2x} ; \quad 0 < x < \infty \\ u\left(\frac{1}{2}t, t\right) &= e^{-t} ; \quad 0 < t < \infty \end{aligned}$$

Ans We have $c = 1$.

For $x > t$, we have d'Alembert's solution

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(e^{-(x+t)} + e^{-(x-t)}\right) + \frac{1}{2}\int_{x-t}^{x+t} e^{-2s} ds \\ &= \frac{1}{2}e^{-(x+t)} - \frac{1}{4}e^{-2(x+t)} + \frac{1}{2}e^{-(x-t)} + \frac{1}{4}e^{-2(x-t)} \end{aligned}$$

In the region $\frac{1}{2}t < x < t$, we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}e^{-(x+t)} - \frac{1}{4}e^{-2(x+t)} + \phi(t-x) \\ e^{-t} &= u\left(\frac{1}{2}t, t\right) = \frac{1}{2}e^{-\frac{3}{2}t} - \frac{1}{4}e^{-3t} + \phi\left(\frac{1}{2}t\right) \\ \phi\left(\frac{1}{2}t\right) &= \frac{1}{4}e^{-3t} - \frac{1}{2}e^{-\frac{3}{2}t} + e^{-t} \\ \phi(t) &= \frac{1}{4}e^{-6t} - \frac{1}{2}e^{-3t} + e^{-2t} \\ \phi(t-x) &= \frac{1}{4}e^{-6(t-x)} - \frac{1}{2}e^{-3(t-x)} + e^{-2(t-x)} \end{aligned}$$

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4. Find particular solutions of the following inhomogeneous wave equations.

- (a) $u_{xx} - c^{-2}u_{tt} = \cos x$
 (b) $u_{xx} - c^{-2}u_{tt} = \sinh t$
 (c) $u_{xx} - c^{-2}u_{tt} = x^2 + t^2$
 (d) $u_{xx} - c^{-2}u_{tt} = (x + t)^2$

Ans (a) Try $u(x, t) = U(x)$. Then

$$U'' = \cos x ; U(x) = -\cos x .$$

(b) Try $u(x, t) = U(t)$. Then

$$U'' = -c^2 \sinh t ; U(t) = -c^2 \sinh t .$$

(c) Try $u(x, t) = U_1(x) + U_2(t)$ Then

$$U_1'' = x^2 ; U_1(x) = \frac{1}{12}x^4$$

$$U_2'' = -c^2 t^2 ; U_2(t) = -\frac{c^2}{12}t^4$$

(d) Either integrate over the triangle, or try $u(x, t) = A(x + t)^4$. Then

$$12A(x + t)^2 - 12Ac^{-2}(x + t)^2 = (x + t)^2 ; A = c^2/(12(c^2 - 1))$$

which gives the solution provided $c \neq 1$.

When $c = 1$, set $\xi = x - t$, $\eta = x + t$,

$$4u_{\xi\eta} = \eta^2$$

$$u = \frac{1}{12}\xi\eta^3 = \frac{1}{12}(x^2 - t^2)(x + t)^2$$

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5. Find the solution of the initial/boundary value problem

$$u_{xx} - \frac{1}{4}u_{tt} = \sin(x + t) ; \quad 0 < x < \infty ; \quad 0 < t < \infty$$

$$u(x, 0) = \cos x ; \quad u_t(x, 0) = \sin x ; \quad 0 < x < \infty$$

$$u_x(0, t) = 0 ; \quad 0 < t < \infty .$$

Ans Firstly, we need to find a particular solution of the inhomogeneous equation.

If we try $U = A \sin(x + t)$,

$$U_{xx} - \frac{1}{4}U_{tt} = -A \sin(x + t) + \frac{1}{4}A \sin(x + t) = -\frac{3}{4}A \sin(x + t)$$

so that $U = -\frac{4}{3} \sin(x + t)$ is a particular solution.

Now setting $u(x, t) = v(x, t) + U$, we have the homogeneous problem

$$\begin{aligned} v_{xx} - \frac{1}{4}v_{tt} &= 0; \quad 0 < x < \infty; 0 < t < \infty \\ v(x, 0) &= \cos x + \frac{4}{3} \sin x \\ v_t(x, 0) &= \sin x + \frac{4}{3} \cos x \\ v_x(0, t) &= \frac{4}{3} \cos t \end{aligned}$$

For $x > 2t$,

$$\begin{aligned} v(x, t) &= \frac{1}{2} (\cos(x + 2t) + \cos(x - 2t)) + \frac{2}{3} (\sin(x + 2t) + \sin(x - 2t)) \\ &\quad + \frac{1}{4} \int_{x-2t}^{x+2t} \left(\sin s + \frac{4}{3} \cos s \right) ds \\ &= \frac{1}{2} (\cos(x + 2t) + \cos(x - 2t)) + \frac{2}{3} (\sin(x + 2t) + \sin(x - 2t)) \\ &\quad - \frac{1}{4} \cos(x + 2t) + \frac{1}{4} \cos(x - 2t) + \frac{1}{3} \sin(x + 2t) - \frac{1}{3} \sin(x - 2t) \\ &= \frac{1}{4} \cos(x + 2t) + \sin(x + 2t) + \frac{3}{4} \cos(x - 2t) + \frac{1}{3} \sin(x - 2t) \end{aligned}$$

For $0 < x < 2t$

$$\begin{aligned} v(x, t) &= \frac{1}{4} \cos(x + 2t) + \sin(x + 2t) + \phi\left(t - \frac{1}{2}x\right) \\ v_x(x, t) &= -\frac{1}{4} \sin(x + 2t) + \cos(x + 2t) - \frac{1}{2} \phi'\left(t - \frac{1}{2}x\right) \\ v_x(0, t) &= -\frac{1}{4} \sin(2t) + \cos(2t) - \frac{1}{2} \phi'(t) = \frac{4}{3} \cos t \\ \phi'(t) &= 2 \cos(2t) - \frac{1}{2} \sin(2t) - \frac{8}{3} \cos(t) \\ \phi(t) &= \sin(2t) + \frac{1}{4} \cos(2t) - \frac{8}{3} \sin(t) + c \\ v(0, 0) &= \cos 0 + \frac{4}{3} \sin 0 = 1 \\ &= \frac{1}{4} \cos 0 + \sin 0 + \phi(0) \\ \phi(0) &= \frac{3}{4}; \quad c = \frac{1}{2} \end{aligned}$$

$$v(x, t) = \frac{1}{4} \cos(x + 2t) + \sin(x + 2t) + \sin(2t - x) + \frac{1}{4} \cos(2t - x) - \frac{8}{3} \sin\left(t - \frac{1}{2}x\right) + \frac{1}{2}$$

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6. Find the solution of the telegraph equation

$$u_{xx} = u_{tt} + 4u_t + 4u$$

with the initial conditions

$$u(x, 0) = e^x ; u_t(x, 0) = e^x .$$

Ans Setting $u(x, t) = e^{-2t}v(x, t)$, we have

$$\begin{aligned} v_{xx} &= v_{tt} \\ v(x, 0) &= u(x, 0) = e^x \\ v_t(x, t) &= 2e^{2t}u(x, t) + e^{2t}u_t(x, t) \\ v_t(x, 0) &= 2e^x + e^x = 3e^x \\ v(x, t) &= \frac{1}{2} (e^{x+t} + e^{x-t}) + \frac{1}{2} \int_{x-t}^{x+t} 3e^s ds \\ v(x, t) &= 2e^{x+t} - e^{x-t} \\ u(x, t) &= 2e^{x-t} - e^{x-3t} \end{aligned}$$

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