MATH 3403

Tutorial Sheet 3

Solutions

1. Reduce the following second order operators to the appropriate canonical form, indicating whether it is elliptic or hyperbolic.

(a) $$u_{xx} + 2u_{xy} + 5u_{yy}$$

(b) $$3u_{xx} + 10u_{xy} + 3u_{yy}$$

(c) $$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

Ans:

1(a) The quadratic equation $X^2 + 2X + 5 = 0$ has the complex conjugate roots $X = -1 \pm 2i$, so that this operator is **elliptic**.

The complex characteristic variable ζ satisfies the equation $\zeta_x + (1 + 2i)\zeta_y = 0$. Hence

$$\frac{dx}{dt} = 1 \ , \ \frac{dy}{dt} = 1 + 2i \ , \ dy = (1 + 2i)dx \ , \ x - y + 2ix = \zeta$$

Taking the real and imaginary parts, we have $\xi = x - y$, $\eta = 2x$.

Under this change of variables

$$u_{xx} + 2u_{xy} + 5u_{yy} \ \equiv \ 4(u_{\xi\xi} + u_{\eta\eta}) \ .$$

1(b) The quadratic equation $3X^2 + 10X + 3 = 0$ has the distinct real roots $X = -3$ and $X = -\frac{1}{3}$. Therefore this operator is **hyperbolic**.

The first characteristic variable ξ can be chosen to satisfy $3\xi_x + \xi_y = 0$. Hence

$$\frac{dx}{dt} = 3 \ , \ \frac{dy}{dt} = 1 \ , \ dx - 3dy = 0 \ , \ x - 3y = \xi$$

The second characteristic variable η satisfies $\eta_x + 3\eta_y = 0$; $\eta = y - 3x$.

Under this change of variables

$$3u_{xx} + 10u_{xy} + 3u_{yy} \ \equiv \ 64u_{\xi\eta} \ .$$

1(c) The quadratic equation associated with this operator is $X^2 + 1/r^2 = 0$. Note that the term $\frac{1}{r}u_r$ is not involved at this stage. It has been included in the operator to simplify the final form. Since the roots of the equation are $X = \pm\frac{i}{r}$, the operator is **elliptic**.

The complex characteristic variable ζ satisfies $\zeta_r + \frac{i}{r}\zeta_\theta = 0$. Hence

$$\frac{dr}{dt} = 1 \ , \ \frac{d\theta}{dt} = \frac{i}{r} \ , \ \frac{dr}{d\theta} = -ir \ , \ r = ae^{-i\theta} \ , \ \zeta = re^{i\theta}$$

Taking the real and imaginary parts we obtain the new variables $x = r\cos\theta$ and $y = r\sin\theta$. Substituting these into the operator we obtain

$$\begin{aligned} u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r \ &\equiv u_{xx}\left(\cos^2\theta + \frac{(-r\sin\theta)^2}{r^2}\right) + 2u_{xy}\left(\cos\theta\sin\theta + \frac{(-r^2\sin\theta\cos\theta)}{r^2}\right) \\ &\quad + u_{yy}\left(\sin^2\theta + \frac{(r\cos\theta)^2}{r^2}\right) + u_x\left(-\frac{1}{r^2}r\cos\theta\right) \\ &\quad + u_y\left(-\frac{1}{r^2}r\sin\theta\right) + \frac{1}{r}(u_x\cos\theta + u_y\sin\theta) \\ &= u_{xx} + u_{yy} \end{aligned}$$

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The original operator is the **Laplacean** operator expressed in polar co-ordinates.

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2. Classify the following operators as to type in the xy plane:

(a) $$x^2u_{xx} + xu_{xy} - yu_{yy}$$

(b) $$u_{xx} + x^2yu_{yy}$$

(c) $$u_{xx} - xyu_{yy}$$

Ans
2(a). The discriminant of the quadratic $\{(b^2 - 4ac)\}$ is $x^2 + 4x^2y$.
Therefore the operator is
hyperbolic when $x^2(1 + 4y) > 0$, $y > -\frac{1}{4}$, $x \neq 0$;
parabolic when $x^2(1 + 4y) = 0$, $y = -\frac{1}{4}$ or $x = 0$;
elliptic when $x^2(1 + 4y) < 0$, $y < -\frac{1}{4}$, $x \neq 0$.

2(b) The discriminant is $-x^2y$. Therefore the operator is
hyperbolic when $x^2y < 0$, $y < 0$, $x \neq 0$;
parabolic when $x^2y = 0$, $x = 0$ or $y = 0$;
elliptic when $x^2y > 0$, $y > 0$, $x \neq 0$.

2(c) The discriminant is xy. Therefore the operator is
hyperbolic when $xy > 0$, ($x > 0$ and $y > 0$) or ($x < 0$ and $y < 0$);
parabolic when $xy = 0$, $x = 0$ or $y = 0$;
elliptic when $xy < 0$, ($x > 0$ and $y < 0$) or ($x < 0$ and $y > 0$). ■

3. Find the general solution of

$$u_{xx} - 5u_{xy} + 6u_{yy} = 0.$$

Ans 3. The quadratic equation is $X^2 - 5X + 6 = 0$, whose roots are $X = 2$ and $X = 3$. This gives characteristic variables ξ, η which satisfy $\xi_x - 2\xi_y = 0$ and $\eta_x - 3\eta_y = 0$. Hence, for ξ,

$$\frac{dx}{dt} = 1 \ , \ \frac{dy}{dt} = -2 \ , \ 2dx + dy = 0 \ , \ \xi = 2x + y$$

and for η,

$$\frac{dx}{dt} = 1 \ , \ \frac{dy}{dt} = -3 \ , \ 3dx + dy = 0 \ , \ \eta = 3x + y$$

Making this change of variables

$$u_{xx} - 5u_{xy} + 6u_{yy} \ \equiv \ -u_{\xi\eta} = 0$$
$$u = f(\xi) + g(\eta) = f(2x + y) + g(3x + y)$$

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