MATH3403

Tutorial Sheet 2

Solutions

Solve the following quasi-linear equations.
Determine the region in which a solution exists in each case.

(1)
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{xy}{u}$$
$$u(1,y) = \sqrt{2y}\ ,\ y > 0$$

Ans: The characteristic equations are

$$\frac{dx}{dt} = x$$
$$\frac{dy}{dt} = y$$
$$\frac{du}{dt} = \frac{xy}{u}$$

The initial data is

$$x = 1\ ;\ y = s\ ;\ u = \sqrt{2s}\ ;\ s > 0$$

The solutions of the first two characteristic equations are

$$x = e^t\ ;\ y = se^t$$

so that the equation for u becomes

$$u\frac{du}{dt} = se^{2t}$$
$$\frac{1}{2}u^2 - s = \frac{1}{2}(se^2t - s)$$
$$u^2 = se^{2t} + s = xy + \frac{y}{x}$$
$$u = \sqrt{\frac{(x^2+1)y}{x}}\ ; x > 0\ ,\ y > 0$$

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(2)
$$u\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x$$
$$u(x,1) = x - 1$$

Ans: The initial data is

$$x = s\ ;\ y = 1\ ;\ u = s - 1$$

The charateristic equations are

$$\frac{dx}{dt} = u$$
$$\frac{dy}{dt} = y$$
$$\frac{du}{dt} = x$$

The second equation gives $y = e^t$, while combining the first and the third gives

$$\frac{d^2x}{dt^2} = x$$
$$x = Ae^t + Be^{-t}$$
$$u = \frac{dx}{dt} = Ae^t - Be^{-t}$$
$$A + B = s \; ; A - B = s - 1$$
$$A = s - \frac{1}{2} \; ; \; B = \frac{1}{2}$$
$$x = se^t - \sinh t \; ; \; u = se^t - \cosh t$$
$$s = e^{-t}(x + \sinh t) = \frac{1}{y}\left(x + \frac{1}{2}\left(y - y^{-1}\right)\right)$$

Substituting into the form for u gives

$$u = x + \frac{1}{2}\left(y - y^{-1}\right) - \frac{1}{2}\left(y + y^{-1}\right)$$
$$= x - \frac{1}{y}$$

which holds for $y > 0$. ★

(3)
$$x(y-u)\frac{\partial u}{\partial x} + y(u-x)\frac{\partial u}{\partial y} = u(x-y)$$
$$u(s,s) = -\frac{2s}{s^2+1} \; , s > 0$$

Ans: From the lecture notes, the characteristic equations have the first integrals

$$x + y + u = a$$
$$xyu = b$$

From the initial data

$$x + y + u = 2s - \frac{2s}{s^2+1} = \frac{2s^3}{s^2+1}$$
$$xyu = -\frac{2s^3}{s^2+1} = -(x+y+u)$$
$$u = -\frac{x+y}{xy+1}$$

Since $s > 0$, the solution is restricted by the condition $xyu < 0$.
Therefore the solution is confined to the octant $x > 0, y > 0, u < 0$. ★