MATH 3403

TUTORIAL SHEET 10

SOLUTIONS

1(a) $(u,v) = \int_0^1 (u'v' + xuv)\,dx$.

$$w_0 = x \;;\; w_0' = 1$$

$$v_1 = x(1-x) \;;\; v_1' = 1-2x$$

$$(w_0, v_1) = \int_0^1 \left[(1-2x) + x^3(1-x)\right]\,dx$$

$$= x - x^2 + \frac{1}{4}x^4 - \frac{1}{5}x^5\Big|_0^1 = \frac{1}{20}$$

$$(v_1, v_1) = \int_0^1 \left[(1-4x+4x^2) + x^3(1-2x+x^2)\right]\,dx$$

$$= x - 2x^2 + \frac{4}{3}x^3 + \frac{1}{4}x^4 - \frac{2}{5}x^5 + \frac{1}{6}x^6\Big|_0^1 = \frac{7}{20}$$

The minimum occurs when

$$\frac{7}{20}a = -\frac{1}{20}$$

$$a = -\frac{1}{7}$$

$$w = \frac{6}{7}x + \frac{1}{7}x^2$$

(b) $(u,v) = \int_0^\pi (u'v' + \sin x\, uv)\,dx$.

$$w_0 = 1 \;;\; w_0' = 0$$

$$v_1 = \sin x \;;\; v_1' = \cos x$$

$$(w_0, v_1) = \int_0^\pi \sin^2 x\,dx = \frac{1}{2}\int_0^\pi (1-\cos 2x)\,dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x\Big|_0^\pi = \frac{1}{2}\pi$$

$$(v_1, v_1) = \int_0^\pi [\cos^2 x + \sin^3 x]\,dx$$

$$= \frac{1}{2}\int_0^\pi (1+\cos 2x)\,dx + \int_0^\pi \sin x(1-\cos^2 x)\,dx$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x - \cos x + \frac{1}{3}\cos^3 x\Big|_0^\pi = \frac{1}{2}\pi + \frac{4}{3}$$

The minimum occurs when

$$\left(\frac{1}{2}\pi + \frac{4}{3}\right)a = -\frac{1}{2}\pi$$

$$a = -\frac{3\pi}{3\pi+8}$$

$$w = 1 - \frac{3\pi}{3\pi+8}\sin x$$

(c) $(u,v) = \int_0^1 \int_0^1 (u_x v_x + u_y v_y + uv)\, dx\, dy$.

$$w_0 = xy \ ; \ w_{0x} = y \ ; \ w_{0y} = x$$

$$v_1 = \sin(\pi x)\sin(\pi y) \ ; v_{1x} = \pi\cos(\pi x)\sin(\pi y) \ ; \ v_{1y} = \pi\sin(\pi x)\cos(\pi y)$$

$$(w_0, v_1) = \pi \left(\int_0^1 \cos\pi x\, dx\right)\left(\int_0^1 y\sin\pi y\, dy\right)$$

$$+\pi\left(\int_0^1 x\sin\pi x\, dx\right)\left(\int_0^1 \cos\pi y\, dy\right) + \left(\int_0^1 x\sin\pi x\, dx\right)\left(\int_0^1 y\sin\pi y\, dy\right)$$

$$\int_0^1 \cos\pi x\, dx = \frac{1}{\pi}\sin\pi x\Big|_0^1 = 0$$

$$\int_0^1 x\sin\pi x\, dx = -\frac{1}{\pi}x\cos\pi x\Big|_0^1 + \frac{1}{\pi}\int_0^1 \cos\pi x\, dx = \frac{1}{\pi}$$

$$(w_0, v_1) = \frac{1}{\pi^2}$$

$$(v_1, v_1) = \pi^2\left(\int_0^1 \cos^2\pi x\, dx\right)\left(\int_0^1 \sin^2\pi y\, dy\right)$$

$$+\pi^2\left(\int_0^1 \sin^2\pi x\, dx\right)\left(\int_0^1 \cos^2\pi y\, dy\right) + \left(\int_0^1 \sin^2\pi x\, dx\right)\left(\int_0^1 \sin^2\pi y\, dy\right)$$

$$= \frac{\pi^2}{4} + \frac{\pi^2}{4} + \frac{1}{4}$$

The minimum occurs when

$$\frac{2\pi^2+1}{4}a = -\frac{1}{\pi^2}$$

$$a = -\frac{4}{\pi^2(2\pi^2+1)}$$

$$w = xy - \frac{4}{\pi^2(2\pi^2+1)}\sin\pi x\,\sin\pi y$$

(d) $(u,v) = \int_0^1 (u'v' + uv)\,dx$.

$$w_0 = x \ ; \ w_0' = 1$$
$$v_n = \sin n\pi x \ ; \ v_n' = n\pi \cos n\pi x$$
$$(w_0, v_n) = \int_0^1 \left[n\pi \cos n\pi x + x \sin n\pi x\right]\,dx$$
$$= \sin n\pi x - \frac{1}{n\pi} x \cos n\pi x\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi x\,dx$$
$$= \frac{(-1)^{n+1}}{n\pi}$$
$$(v_n, v_m) = nm\pi^2 \int_0^1 \cos n\pi x\ \cos m\pi x\,dx + \int_0^1 \sin n\pi x\ \sin m\pi x\,dx = 0$$
$$(v_n, v_n) = n^2\pi^2 \int_0^1 \cos^2 n\pi x\,dx + \int_0^1 \sin^2 n\pi x\,dx = \frac{n^2\pi^2 + 1}{2}$$
$$w = x + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n^2\pi^2+1)} \sin n\pi x$$

2.

$$\int_{-1}^1 (u')^2\,dx = 2\int_0^1 \alpha^2 x^{2\alpha-2}\,dx = \frac{2\alpha^2}{2\alpha-1}$$
$$\int_{-1}^1 u^2\,dx = 2\int_0^1 (1 - 2x^\alpha + x^{2\alpha})\,dx = 2 - \frac{4}{\alpha+1} + \frac{2}{2\alpha+1}$$
$$= \frac{2 + 6\alpha + 4\alpha^2 - 4 - 8\alpha + 2 + 2\alpha}{(1+\alpha)(1+2\alpha)}$$

The Rayleight quotient is

$$\frac{2\alpha^2}{2\alpha-1}\frac{(1+\alpha)(1+2\alpha)}{4\alpha^2} = \frac{(1+\alpha)(1+2\alpha)}{2(2\alpha-1)} = \frac{1}{2}\left(\alpha + 2 + \frac{3}{2\alpha-1}\right)$$

The minimum occurs when

$$1 - \frac{6}{(2\alpha-1)^2} = 0$$
$$(2\alpha^2 - 1)^2 = 6$$
$$\alpha = \frac{1+\sqrt{6}}{2}$$
$$\mu^2 = \frac{1}{2}\left(\frac{1+\sqrt{6}}{2} + 2 + \frac{3}{\sqrt{6}}\right) = \frac{1}{4}(5 + 2\sqrt{6})$$
$$\mu = \frac{1}{2}(\sqrt{2} + \sqrt{3})$$

3. The eigenfunctions for

$$u_{xx} + u_{yy} = \mu^2 u$$

on the square $0 < x < \pi$, $0 < y < \pi$ with Dirichlet boundary data are

$$u_{mn} = \sin(mx)\sin(ny)\ ,$$

and the corresponding eigenvalues are $\mu^2 = m^2 + n^2$. The separated solutions of the wave equation have the form

$$\sin(mx)\sin(ny)(a_{mn}\cos(\sqrt{m^2+n^2}t) + b_{mn}\sin(\sqrt{m^2+n^2}t))\ .$$

The initial condition $u_t(x, y, 0) = 0$ gives $b_{mn} = 0$ for all m, n. The initial condition $u(x, y, 0) = \sin^3 x \sin^3 y$ which can be written as

$$\frac{9}{16}\sin x \sin y - \frac{3}{16}\sin 3x \sin y - \frac{3}{16}\sin x \sin 3y + \frac{1}{16}\sin 3x \sin 3y\ .$$

Therefore, $a_{11} = \frac{9}{16}$, $a_{31} = a_{13} = -\frac{3}{16}$, and $a_{33} = \frac{1}{16}$. The solution is therefore

$$\begin{aligned} u(x, y, t) = & \frac{9}{16}\sin x \sin y \cos(\sqrt{2}t) + \frac{1}{16}\sin 3x \sin 3y \cos(3\sqrt{2}t) \\ & -\frac{3}{16}(\sin x \sin 3y + \sin 3x \sin y)\cos(\sqrt{10}t) \end{aligned}$$

4. (a) If $\nabla^2\phi = \lambda P\phi$, then

$$\begin{aligned} \iiint_{\mathcal{D}} (\nabla\phi)^2\, dV &= \iint_{\partial\mathcal{D}} \phi\frac{\partial\phi}{\partial n}\, dS - \iiint_{\mathcal{D}} \phi\nabla^2\phi\, dV \\ \iiint_{\mathcal{D}} (\nabla\phi)^2 + \lambda P\phi^2)\, dV &= \iint_{\partial\mathcal{D}} \phi\frac{\partial\phi}{\partial n}\, dS = 0 \end{aligned}$$

Therefore if $\lambda > 0$, $\phi = 0$ in $\mathcal{D}$.

(b) If $\lambda = -\mu^2$, then

$$\begin{aligned} \iiint_{\mathcal{D}} (\nabla\phi)^2 - \mu^2 P\phi^2)\, dV = 0 \\ \mu^2 \iiint_{\mathcal{D}} P\phi^2\, dV = \iint_{\mathcal{D}} (\nabla\phi)^2\, dV \end{aligned}$$

(c)

$$\begin{aligned} \iiint_{\mathcal{D}} \nabla\phi_1\nabla\phi_2\, dV &= -\iiint_{\mathcal{D}} \phi_1\nabla^2\phi^2 \\ &= \mu_2^2 \iiint_{\mathcal{D}} P\phi_1\phi_2\, dV \\ &= \mu_1^2 \iiint_{\mathcal{D}} P\phi_1\phi_2\, dV \text{ by symmentry} \end{aligned}$$

therefore, if $\mu_1^2 \neq \mu_2^2$ both the left and right hand sides of the above equation must be zero.

(d) Since the integrals of the cross terms are zero from (c),

$$\iiint_{\mathcal{D}} (\nabla f)^2 \, dV = \sum_{n=1}^{\infty} a_n^2 \iiint_{\mathcal{D}} (\nabla \phi_n)^2 \, dV$$
$$= \sum_{n=1}^{\infty} a_n^2 \mu_n^2 \iiint_{\mathcal{D}} P \phi_n^2 \, dV$$
$$\geq \sum_{n=1}^{\infty} a_n^2 \mu_1^2 \iiint_{\mathcal{D}} P \phi_n^2 \, dV = \mu_1^2 \iiint_{\mathcal{D}} f^2 \, dV$$

(e) See lecture notes.

5.

$$\nabla^2 f = \frac{1}{r} \frac{d}{dr} \left(r \frac{df}{dr} \right) = -\alpha^2 r^{\alpha - 2}$$
$$\iint (\nabla f)^2 \, dS = - \iint f \nabla^2 f \, dS$$
$$= \alpha^2 \int_0^{2\pi} d\theta \int_0^1 r(1 - r^\alpha) r^{\alpha - 2} \, dr$$
$$= 2\pi\alpha^2 \left[\frac{1}{\alpha} r^\alpha - \frac{1}{2\alpha} r^{2\alpha} \right]_0^1 = \pi\alpha$$
$$\iint r f^2 \, dS = \int_0^{2\pi} d\theta \int_0^1 r^2 (1 - r^\alpha)^2 \, dr$$
$$= 2\pi \left[\frac{1}{3} - \frac{2}{\alpha + 3} + \frac{1}{2\alpha + 3} \right] = \frac{4\pi\alpha^2}{3(\alpha + 3)(2\alpha + 3)}$$

The Rayleight quotient is

$$\frac{3(\alpha + 3)(2\alpha + 3)}{4\alpha} = \frac{3}{2}\alpha + \frac{27}{4} + \frac{27}{4}\alpha^{-1}$$

which has a minimum when

$$0 = \frac{3}{2} - \frac{27}{4}\alpha^{-2}$$
$$\alpha^2 = \frac{9}{2}$$
$$\frac{3}{2}\alpha + \frac{27}{4} + \frac{27}{4}\alpha^{-1} = \frac{9}{\sqrt{2}} + \frac{27}{4} = \frac{9}{4}(1 + \sqrt{2})^2$$