MATH 3403

TUTORIAL SHEET 1

SOLUTIONS

1. Determine the characteristic through the point $(1, 1)$ for the linear operator

$$(x + 2)u_x + 2yu_y \ .$$

Ans: The characteristic equations are

$$\frac{dx}{dt} = (x + 2) \ ; x(0) = 1$$
$$\frac{dy}{dt} = 2y \ ; y(0) = 1$$

for which the solutions are

$$x + 2 = 3e^t$$
$$y = e^{2t}$$

Therefore the characteristic is the branch of the parabola

$$y = \left(\frac{x+2}{3}\right)^2$$

for $x > -2$. ■

If

$$(x + 2)u_x + 2yu_y = 2u \ ,$$

and $u(1, 1) = 1$, determine $u(4, 4)$.

Ans: Along the characteristic through $(1, 1)$, if $e^t = 2$, then $x = 4$ and $y = 4$. Therefore the value of $u(4, 4)$ is determined by the value of $u(1, 1)$.

We have

$$\frac{du}{dt} = 2u \ ; u(0) = 1$$

so that

$$u = e^{2t} = y$$

and $u(4, 4) = 4$. ■

2. Determine the characteristic through the point $(1, -1)$ for the linear operator

$$x^2u_x - y^2u_y \ .$$

Ans: The characteristic equations are

$$\frac{dx}{dt} = x^2 \ ; \ x(0) = 1$$
$$\frac{dy}{dt} = -y^2 \ ; y(0) = -1$$

Therefore

$$\begin{aligned}\frac{dx}{x^2} &= dt \\ 1 - \frac{1}{x} &= t \\ -\frac{dy}{y^2} &= dt \\ 1 + \frac{1}{y} &= t \\ 1 - \frac{1}{x} &= 1 + \frac{1}{y} \\ x + y &= 0\end{aligned}$$

and the characteristic is the ray $x = -y$ for $x > 0$. ■

If

$$x^2 u_x - y^2 u_y = u \ ,$$

and $u(1, -1) = 1$, determine $u(2, -2)$.

Ans: The point $(2, -2)$ lies on the characteristic for $t = \frac{1}{2}$.

If

$$\frac{du}{dt} = u \ , \ u(0) = 1$$

then

$$u = e^t \ , \text{ and } u(2, -2) = e^{1/2}$$

■

Find the solutions of the following equations, indicating the regions in which the solutions are defined.

3. $u_x + u_y = 1; \quad u = \phi(s) \quad \text{on} \quad x = s, y = 2s.$

Ans: The characteristic equations are

$$\begin{aligned}\frac{dx}{dt} &= 1 \ ; \ x(0) = s \\ \frac{dy}{dt} &= 1 \ ; \ y(0) = 2s \\ \frac{du}{dt} &= 1 \ ; \ u(0) = \phi(s)\end{aligned}$$

so that

$$\begin{aligned}x &= t + s \\ y &= t + 2s \\ s &= y - x \\ t &= 2x - y \\ u &= t + \phi(s) \\ &= 2x - y + \phi(y - x)\end{aligned}$$

Since the characteristics cover the whole plane, the solution is defined for all x and y. ■

4. $x^2u_x - y^2u_y = 0; \quad u(1,y) = F(y).$

Ans: We can parametrise the initial data as $x = 1$, $y = s$, $u = F(s)$ for all s. The characteristic equations are

$$\begin{aligned}\frac{dx}{dt} &= x^2 \ ; \ x(0) = 1\\ \frac{dy}{dt} &= -y^2 \ ; y(0) = s\\ \frac{du}{dt} &=0 \ ; u(0) = F(s)\end{aligned}$$

so that

$$\begin{aligned}1 - \frac{1}{x} &= t\\ x &= \frac{1}{1-t} \ , \ t < 1\\ \frac{1}{y} - \frac{1}{s} &= t \ , \ s \neq 0\\ y &= \frac{s}{1+st} \begin{cases} t < -\frac{1}{s} \ , s < 0\\ -\frac{1}{s} < t \ , s > 0\end{cases}\\ y &= 0 \ ; \ s = 0\end{aligned}$$

Eliminating t between the equations for x and y gives

$$y = \frac{sx}{(1+s)x - s} \ ; \ s = \frac{xy}{x + y - xy}$$

which represent hyperbolae in general.

The characteristics are:

the branches of the hyperbolae to the right of the asymptote $x = s/(1+s)$ and above the asymptote $y = s/(1+s)$ for $s > 0$;

the positive x-axis for $s = 0$;

the portions of the hyperbolae for $0 < x < \infty$ for $-1 \leq s < 0$;

and the portions of the hyperbolae for $0 < x < \frac{s}{s+1}$ for $s < -1$.

The region covered by the characteristics is bounded to the right and above by the parabola $y = x/(x-1)$, and bounded on the left by the y-axis for $y > 0$ and by the arc $y = \frac{x}{x-1}$ for $y < 0$. The solution is defined in this region.

The solution for u is

$$u = F(s) = F\left(\frac{xy}{x+y-xy}\right) \ .$$

■

5. $xu_x + yu_y = 3; \quad u(1,y) = \log y \, , \quad y > 1.$

Ans: We can parametrise the initial data as $x = 1$, $y = s$, $u = \log s$, for $s > 1$. The characteristic equations are

$$\begin{aligned}\frac{dx}{dt} &= x \ ; \ x(0) = 1\\ \frac{dy}{dt} &= y \ ; \ y(0) = s\\ \frac{du}{dt} &= 3 \ ; \ u(0) = \log s\end{aligned}$$

which gives

$$x = e^t \ ; \ t = \log x$$
$$y = se^t \ ; \ s > 1 \ ; \ s = y/x$$

so that the characteristics are rays emanating from the origin and covering the octant bounded by the line $y = x$ and the y-axis. The solution is defined in this region.

The solution is

$$\begin{aligned} u &= 3t + \log s \\ &= 3\log x + \log(y/x) \\ &= \log(x^2 y) \ ; \ y > x > 0 \end{aligned}$$

■

6. $yu_x + xu_y = u; \quad u(x,0) = e^{-x}$.

Ans: We can parametrise the initial data as $x = s$, $y = 0$, $u = e^{-s}$.

The characteristic equations are

$$\begin{aligned} \frac{dx}{dt} &= y \ ; \ x(0) = s \\ \frac{dy}{dt} &= x \ ; \ y(0) = 0 \\ \frac{du}{dt} &= u \ ; \ u(0) = e^{-s} \end{aligned}$$

If we differentiate the first equation we obtain

$$\frac{d^2x}{dt^2} = \frac{dy}{dt} = x \ ; \ x(0) = s \ ; x'(0) = y(0) = 0$$

which gives

$$x = \frac{s}{2}\left(e^t + e^{-t}\right) = s\cosh t$$

and

$$y = \frac{dx}{dt} = s\sinh t$$

The characteristics are the hyperbolic arcs $x^2 - y^2 = s^2$, which cover the region $|y| < |x|$ in the $x - y$ plane.

The solution for u is

$$u = e^{-s}e^t \ .$$

We have $x + y = se^t$, and $s = x\sqrt{1 - y^2/x^2}$.

Therefore

$$u = \frac{x+y}{x\sqrt{1-y^2/x^2}} \exp(-x\sqrt{1-y^2/x^2}) \ .$$

■